Control Systems Stability and Robust Control

Lecture #1

Introduction

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Nonlinear State Model

An n-dimensional system takes the following form:

$$\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, \dots, x_n, u_1, \dots, u_m) \\
\dot{x}_2 &= f_2(t, x_1, \dots, x_n, u_1, \dots, u_m) \\
&\vdots \\
\dot{x}_n &= f_n(t, x_1, \dots, x_n, u_1, \dots, u_m)
\end{aligned}$$

x_i: state variables; u_i: input variables; f_i: nonlinear functions

$\dot{x}_i$: derivative of x_i

Nonlinear State Model

Define

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad f(t,x,u) = \begin{bmatrix} f_1(t,x,u) \\ f_2(t,x,u) \\ \vdots \\ f_n(t,x,u) \end{bmatrix}$$

Compact form

$$\dot{x} = f(t,x,u)$$

Unforced state equation

$$\dot{x} = f(t,x)$$

Nonlinear State-Output Model

When there exist outputs, the system takes the following form

$$\dot{x} = f(t, x, u) \qquad \text{(state equation)}$$
$$y = h(t, x, u) \qquad \text{(output equation)}$$

Special cases

$$\dot{x} = A(t)x + B(t)u$$
$$y = C(t)x + D(t)u$$

Autonomous vs. Nonautonomous

Autonomous (Time-Invariant) Systems

$$\dot{x} = f(x)$$

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}$$

Nonautonomous (Time-Varying) Systems

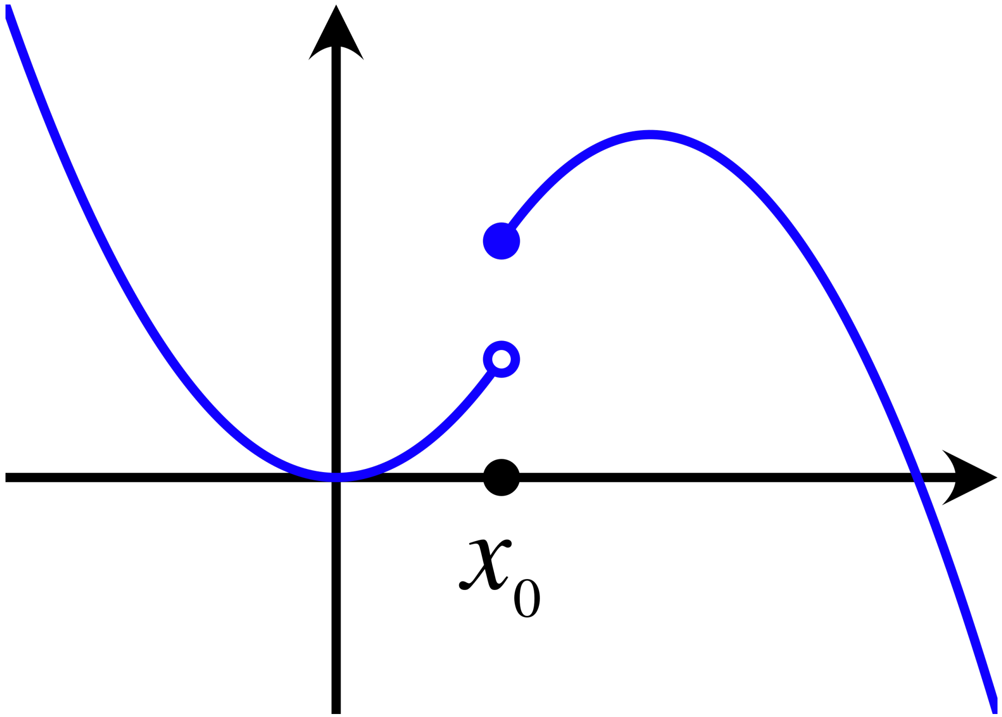
$$\dot{x} = f(x,t)$$

$$\begin{aligned} \dot{x} &= f(x,u,t) \\ y &= h(x,u,t) \end{aligned}$$

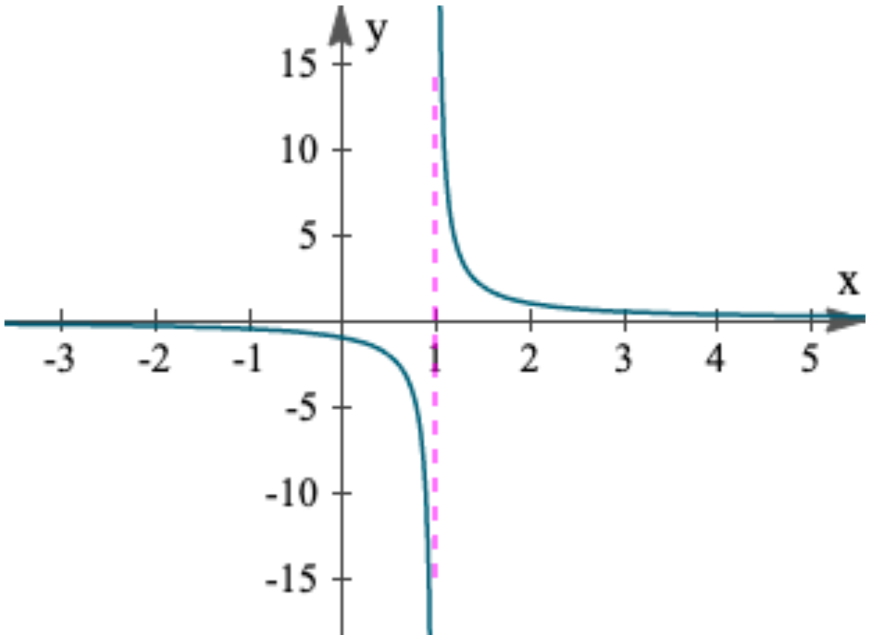
An autonomous system has a time-invariance property with respect to shifting the initial time from t_0 to $t_0 + a$, provided the input waveform is applied form $t_0 + a$ rather than t_0.

Piecewise Continuous Function

$f(t)$ is piecewise continuous on an interval $J \subset R$ if for every bounded subinterval $J_0 \subset J$, f is continuous in t for all $t \in J_0$, except, possibly, at a finite number of points where f might have finite-jump discontinuity.



Piecewise-continuous



Not piecewise-continuous

Lipschitz Condition

Locally Lipschitz

$f(x)$ is locally Lipschitz at a point x_0 if there is a neighborhood $N(x_0, r) = \{x \in R^n \mid ||x - x_0|| < r\}$ where $f(x)$ satisfies the Lipschitz condition

$$||f(x) - f(y)|| \leq L||x - y||, \qquad L > 0$$

We call L a Lipschitz constant.

When $n = 1$, the Lipschitz condition reduces to

$$\frac{|f(y) - f(x)|}{|y - x|} \leq L$$

Globally Lipschitz

A function $f(x)$ is locally Lipschitz on a domain $D \subset R^n$ if it is locally Lipschitz at every point $x_0 \in D$. When $D = R^n$, the function is globally Lipschitz.

Continuously Differentiable → Lipschitz → Continuous

Example

The function $f(x) = x^{\frac{1}{3}}$ is not locally Lipschitz at $x = 0$ since

$$f'(x) = \left(\frac{1}{3}\right) x^{-\frac{2}{3}} \to \infty, \text{ as } x \to 0$$

Will a discontinuous function be Lipschitz? No.

Local Existence and Uniqueness of Solutions

Solutions of a state equation

$$\dot{x} = f(t, x)$$

Local existence and uniqueness

Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x at x_0, for all $t \in [t_0, t_1]$. Then, there is $\delta > 0$ such that the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_0 + \delta]$.

- $\dot{x} = x^{\frac{1}{3}}$ has $x(t) = \left(\frac{2t}{3}\right)^{\frac{3}{2}}$ and $x(t) \equiv 0$ as two different solutions when the initial state is $x(0) = 0$.

Global Existence and Uniqueness of Solutions

Via globally Lipschitz condition

Let $f(t, x)$ be piecewise continuous in t and globally Lipschitz in x for all $t \in [t_0, t_1]$. Then, the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_1]$.

Via locally Lipschitz condition

Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x for all $t \geq t_0$ and all x in a domain $D \subset R^n$. Let W be a compact subset of D, and suppose that every solution of $\dot{x} = f(t, x)$, $x(t_0) = x_0$ with $x_0 \in W$ lies entirely in W. Then, there is a unique solution that is defined for all $t \geq t_0$.

- For Euclidean space, a compact set $\leftrightarrow$ bounded and closed set.

Example

- Consider the following system

$$\dot{x} = -x^3 = f(x)$$

$f(x)$ is locally Lipschitz on R, but not globally Lipschitz because $f'(x) = -3x^2$ is not globally bounded.

- If $x(t)$ is positive, the derivative $\dot{x}(t)$ will be negative. Similarly, if $x(t)$ is negative, the derivative $\dot{x}(t)$ will be positive.
- Therefore, starting from any initial condition $x(0) = a$, the solution cannot leave the compact set $\{x \in R \mid |x| \leq |a|\}$. Thus, the equation has a unique solution for all $t \geq 0$.

Change of Variables

Map: $z = T(x)$ Inverse map: $x = T^{-1}(z)$

- a map $T(x)$ is invertible over its domain D if there is a map $T^{-1}(z)$ such that $x = T^{-1}(z)$ for all $z \in T(D)$
- A map $T(x)$ is a diffeomorphism if $T(x)$ and $T^{-1}(x)$ are continuously differentiable.
- $T(x)$ is a local diffeomorphism at x_0 if there is a neighborhood N of x_0 such that T restricted to N is a diffeomorphism on N
- $T(x)$ is a global diffeomorphism if it is a diffeomorphism on R^n and $T(R^n) = R^n$.

Jacobian Matrix

Jacobian matrix

$$\frac{\partial T}{\partial x} = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \cdots & \frac{\partial T_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial T_n}{\partial x_1} & \cdots & \frac{\partial T_n}{\partial x_n} \end{bmatrix}$$

- The continuously differentiable map $z = T(x)$ is a local diffeomorphism at x_0 if the Jacobian matrix $\left[\frac{\partial T}{\partial x}\right]$ is nonsingular at x_0.
- It is a global diffeomorphism if and only if $\left[\frac{\partial T}{\partial x}\right]$ is nonsingular for all $x \in R^n$ and T is proper; that is, $\lim_{||x|| \to \infty} ||T(x)|| = \infty$.

Equilibrium Points

- For the autonomous system $\dot{x} = f(x)$, the equilibrium points are the real solutions of the equation $f(x^*) = 0$.

- A point $x = x^*$ in the state space is said to be an equilibrium point of $x^* = f(t, x)$ if $f(t, x^*) \equiv 0, \forall\, t \geq t_0$.

- An equilibrium point could be isolated; that is, there are no other equilibrium points in its vicinity, or there could be a continuum of equilibrium points.

Equilibrium Points

- A linear system $\dot{x} = Ax$ can have an isolated equilibrium point at $x = 0$ (if A is nonsingular) or a continuum of equilibrium points in the null space of A (if A is singular).
- A linear system cannot have multiple isolated equilibrium points. Can you prove that?
- A nonlinear state equation can have multiple isolated equilibrium points. For example, the state equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -a \sin x_1 - b x_2$$

has equilibrium points at $(x_1 = n\pi, \ x_2 = 0)$ for $n = 0,1,2,\ldots$