# Nonlinear Control
# Lecture # 10
# Input-Output Stability

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# Input-Output Models

- Consider the input-output model

$$y = Hu$$

where $u(t)$ and $y(t)$ are both piecewise continuous function of $t$ and $H$ is an operator.

- Norm of a signal $||u||$

$$||u|| \geq 0 \text{ and } ||u||=0 \Leftrightarrow u = 0$$

$$||au||=\alpha||u|| \text{ for any } \alpha>0$$

$$\text{Triangle Inequality: } ||u_1 + u_2|| \leq ||u_1|| + ||u_2||$$

# $\mathcal{L}_p$ spaces

- $\mathcal{L}_p$: $\|u\|_{\mathcal{L}_p} = \left(\int_0^\infty \|u(t)\|^p \, dt\right)^{\frac{1}{p}} < \infty, 1 < p < \infty$
- $\mathcal{L}_2$: $\|u\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty u^\top(t)u(t) \, dt} < \infty$
- $\mathcal{L}_\infty$: $\|u\|_{\mathcal{L}_\infty} = \sup_{t \geq 0} \|u(t)\| < \infty$
- $\mathcal{L}_p^m$: $p$ is the type of $p$-norm used to define the space and $m$ is the dimension of $u$.

# Extended Space

- $\mathcal{L}_e = \{u \mid u_\tau \in \mathcal{L}, \forall\, \tau \in [0,\infty)\}$

  where $u_\tau$ is a truncation of u: $u_\tau(t) = \begin{cases} u(t) & , 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$

- $\mathcal{L}_e$ is a linear space and $\mathcal{L} \subset \mathcal{L}_e$

- Example

$$u(t) = t, \qquad u_\tau(t) = \begin{cases} t\,, & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$$

$$u \notin \mathcal{L}_\infty \quad \text{but} \quad u_\tau \in \mathcal{L}_{\infty e}$$

# Causality

- A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is causal if the value of the output $(Hu)(t)$ at any time t depends only on the values of the input up to time $t$

$$(Hu)_\tau = (Hu_\tau)_\tau$$

- A scalar continuous function $g(r)$, defined for $r \in [0, a)$, is a gain function if it is nondecreasing and $g(0) = 0$.

- A class $K$ function is a gain function but not the other way around. By not requiring the gain function to be strictly increasing we can have $g = 0$.

# $\mathcal{L}$ Stability

- A mapping H : $\mathcal{L}_e^m \longrightarrow \mathcal{L}_e^q$ is $\mathcal{L}$ stable if there exist a gain function g, defined on $[0,\infty)$, and a nonnegative constant β such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \le g(\|u_\tau\|_{\mathcal{L}}) + \beta, \quad \forall \mathrm{u} \in \mathcal{L}_e^m \ \ and \ \ \tau \in [0,\infty)$$

- It is finite-gain $\mathcal{L}$ stable if there exist nonnegative constants $\gamma$ and β such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \le \gamma \|u_\tau\|_{\mathcal{L}} + \beta, \quad \forall \mathrm{u} \in \mathcal{L}_e^m \ \ and \ \ \tau \in [0,\infty)$$

In this case, we say that the system has $\mathcal{L}$ gain γ.

- The bias term β is included in the definition to allow for systems where $Hu$ does not vanish at $\mathrm{u} = 0$.

# Example

- Consider the memoryless function $y = h(u)$.
- Suppose $|h(u)| \leq a + b|u|$, ∀ u ∈ R. Then, the system is finite-gain $\mathcal{L}_\infty$ stable with $\beta = a$ and $\gamma = b$.
- If $a = 0$, then for each $p \in [1, \infty)$

$$\int_0^\infty |h(u(t))|^p \, dt \leq (b)^p \int_0^\infty |u(t)|^p \, \mathrm{d}t$$

The system is finite-gain $\mathcal{L}_P$ stable with β = 0 and γ = b.

- Prove $h(u) = u^2$ is $\mathcal{L}_\infty$ but not finite-gain $\mathcal{L}_\infty$ stable.

$h$ is $\mathcal{L}_\infty$ stable with zero bias and g$(r) = r^2$. It is not finite-gain $\mathcal{L}_\infty$ stable because $|h(u)| = u^2$ cannot be bounded by $\gamma|u| + \beta$ for all u ∈ R.

# Small-signal $\mathcal{L}$ Stability

Definition

A mapping H :$\mathcal{L}_e^m \rightarrow \mathcal{L}_e^q$ is small-signal $\mathcal{L}$ stable (respectively, small-signal finite-gain $\mathcal{L}$ stable) if there is a positive constant $r$ such that the condition for $\mathcal{L}$ stability (respectively , finite-gain $\mathcal{L}$ stability ) is satisfied for all $u \in \mathcal{L}_e^m$ with

$$\sup_{0 \le t \le \tau} \|u(t)\| \le r.$$

# Example

Consider the system

$$y = \tan u$$

The output y(t) is defined only when the input signal is restricted to $|u(t)| < \pi/2$ for all $t \geq 0$

$$u(t) \in \left\{|u| \leq r \leq \frac{\pi}{2}\right\} \Rightarrow |y| \leq \left(\frac{\tan r}{r}\right)|u|$$

$$\|y\|_{\mathcal{L}_p} \leq \left(\frac{\tan r}{r}\right)\|u\|_{\mathcal{L}_p}, \qquad p \in [1, \infty)$$

$y$ is small-signal finite-gain $\mathcal{L}_p$ stable.

# Lyapunov Theorem for $\mathcal{L}$ Stability

Consider the system:

$$\dot{x} = f(x,u), \quad y = h(x,u), 0 = f(0,0), 0 = h(0,0)$$

**Theorem**

Suppose, $\forall \|x\| \le r, \forall \|u\| \le r_u$

$$C_1\|x\|^2 \le V(x) \le C_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x,0) \le -C_3\|x\|^2, \qquad \left\|\frac{\partial V}{\partial x}\right\| \le C_4\|x\|$$

$$\|f(x,u) - f(x,0)\| \le L\|u\|, \qquad \|h(x,u)\| \le \eta_1\|x\| + \eta_2\|u\|$$

Then, for each $x_0$ with $\|x_0\| \le r\sqrt{C_1/C_2}$ ,the system is small-signal finite-gain $\mathcal{L}_P$ stable for each p$\in [1, \infty)$. It is finite-gain $L_p$ stable $\forall\ x_0 \in R^n$ if the assumptions hold globally [See you textbook for $\beta$ *and* $\gamma$ ] .

# Example

- Consider the system

$$\dot{x} = -x - x^3 + u, \qquad y = \tanh x + u$$

$$V = \frac{1}{2}x^2 \Rightarrow \dot{V} = x(-x - x^3) \leq -x^2$$

$$c_1 = c_2 = \frac{1}{2}, \qquad c_3 = c_4 = 1, \qquad L = \eta_1 = \eta_2 = 1$$

- Finite-gain $L_P$ stable for each x(0) ∈ R and each p ∈[1,∞]

# ISS -> $L_\infty$ Stable

Theorem

Suppose that, for all $(x, u)$, $f$ is locally Lipschitz and $h$ is continuous and satisfies

$$\|h(x,u)\| \le g_1(\|x\|) + g_2(\|u\|) + \eta, \eta \ge 0$$

For some gain functions $g_1, g_2$. If $\dot{x} = f(x,u)$ is ISS, then for each $x(0) \in R^n$, the system

$$\dot{x} = f(x,u), \qquad y = h(x,u)$$

is $\mathcal{L}_\infty$ stable.

# Example

- Consider the system

$$\dot{x} = -x - 2x^3 + (1 + x^2)u^2, \qquad y = x^2 + u$$

- The system is ISS from Example 4.13.
- Let

$$g_1(r) = r^2, \quad g_2(r) = r, \quad \eta = 0$$

- The system is $\mathcal{L}_\infty$ stable