Nonlinear Control

Lecture #11

Stability of Feedback Systems

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Passivity Theorems

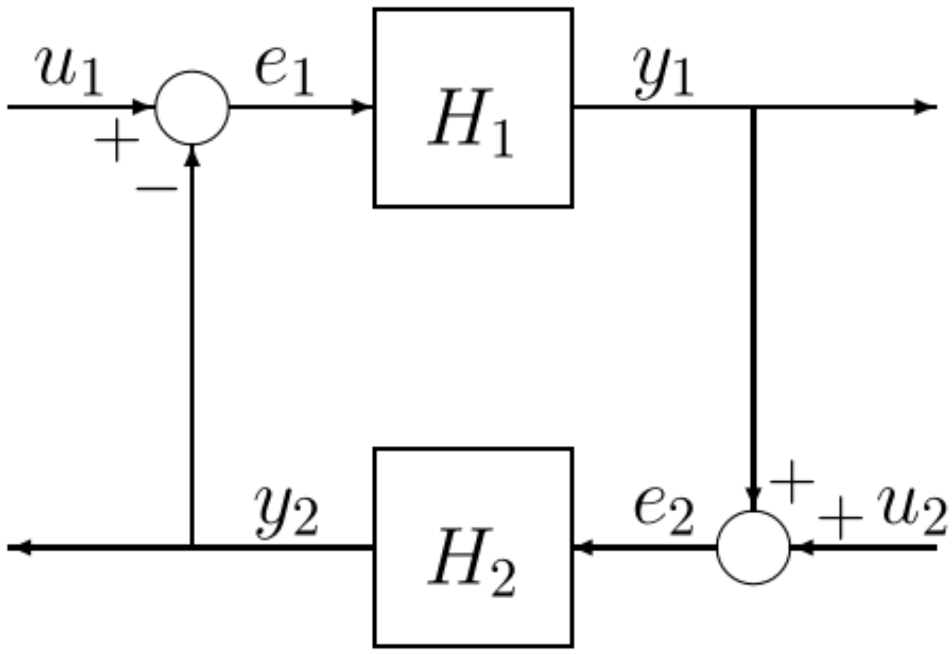

Figure 1: Feedback connection.

Passivity Theorems

- Each of the feedback components H_1 and H_2 is either a time-invariant dynamical system represented by the state model

$$\dot{x}_i = f_i(x_i, e_i), \qquad y_i = h_i(x_i, e_i) \qquad (7.1)$$

- or a (possibly time-varying) memoryless function represented by

$$y_i = h_i(t, e_i) \qquad (7.2)$$

Passivity Theorem

Theorem 7.1

The feedback connection of two passive systems is passive.

Proof

- Let $V_1(x_1)$ and $V_2(x_2)$ be the storage functions for H_1 and H_2. Then, $e_i^T y_i \geq \dot{V}_i$.
- From the feedback connection of Figure 1,we see that

$$\begin{aligned} e_1^T y_1 + e_2^T y_2 &= (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 \\ &= u_1^T y_1 + u_2^T y_2 \end{aligned}$$

  Hence,

$$u^T y = u_1^T y_1 + u_2^T y_2 \geq \dot{V}_1 + \dot{V}_2 = \dot{V}$$

- With $V(x) = V_1(x_1) + V_2(x_2)$ as the storage function for the feedback connection, we obtain $u^T y \geq \dot{V}$.

Asymptotic Stability for Two Dynamical Systems

Theorem 7.2

Consider the feedback connection of two dynamical systems. When $u = 0$, the origin of the closed-loop system is asymptotically stable if one of the following conditions is satisfied:

- both feedback components are strictly passive;
- both feedback components are output strictly passive and zero-state observable;
- one component is strictly passive and the other one is output strictly passive and zero-state observable.

Furthermore, if the storage function for each component is radially unbounded, the origin is globally asymptotically stable.

Proof

Let $V_1(x_1)$ and $V_2(x_2)$ be the storage functions for H_1 and H_2. Take $V(x) = V_1(x_1) + V_2(x_2)$ as a Lyapunov function candidate.

- $u = 0 \Rightarrow \dot{V} \leq u^T y - \psi_1(x_1) - \psi_2(x_2) = -\psi_1(x_1) - \psi_2(x_2)$

  Asymptotically stable.
- $\dot{V} \leq -y_1^T \rho_1(y_1) - y_2^T \rho_2(y_2),\ \ y_i^T \rho_i(y_i) > 0$ for all $y_i \neq 0$.

  $\dot{V} = 0 \Rightarrow y = 0$. Note that $y_2 \equiv 0 \Rightarrow e_1 \equiv 0$. By zero-state-observability, we have $y_1 \equiv 0 \Rightarrow x_1 \equiv 0$. Asymptotically stable.
- H_1 is strictly passive. $\dot{V} \leq -\psi(x_1) - y_2^T \rho_2(y_2)$. $\dot{V} = 0 \Rightarrow x_1 = 0, y_2 = 0$. $y_2(t) \equiv 0 \Rightarrow e_1(t) \equiv 0 (\& x_1(t) \equiv \quad 0) \Rightarrow y_1(t) \equiv 0 \Rightarrow e_2(t) \equiv 0$, by zero-state observability of H_2, $y_2(t) \equiv 0 \Rightarrow x_2(t) \equiv 0$. The origin is asymptotically stable.

Example 7.1

Consider the feedback connection of

$$H_1: \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -ax_1^3 - kx_2 + e_1 \\ y_1 = x_2 \end{cases} \qquad H_2: \begin{cases} \dot{x}_3 = x_4 \\ \dot{x}_4 = -bx_3 - x_4^3 + e_2 \\ y_2 = x_4 \end{cases}$$

where a, b, and k are positive constants. Use $V_1 = \frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2$ and $V_2 = \frac{1}{2}bx_3^2 + \frac{1}{2}x_4^2$ to show that the unforced system is globally asymptotically stable.

Using $V_1 = \frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2$ as a storage function for H_1, we obtain

$$\dot{V}_1 = ax_1^3x_2 - ax_1^3x_2 - kx_2^2 + x_2e_1 = -ky_1^2 + y_1e_1$$

Hence, H_1 is output strictly passive. Besides, with $e_1 = 0$, we have

Example 7.1

$$y_1(t) \equiv 0 \iff x_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

which shows that H_1 is zero-state observable.

Using $V_2 = \frac{1}{2}bx_3^2 + \frac{1}{2}x_4^2$ as a storage function for H_2, we obtain

$$\dot{V}_2 = \mathrm{b}x_3x_4 - bx_3x_4 - x_4^4 + x_4e_2 = -y_2^4 + y_2e_2$$

Therefore, H_2 is output strictly passive. With $e_2 = 0$, we have

$$y_2(t) \equiv 0 \iff x_4(t) \equiv 0 \Rightarrow x_3(t) \equiv 0$$

which shows that H_2 is zero-state observable. The fact that V_1 and V_2 are radially unbounded, we conclude that the origin is globally asymptotically stable.

Example 7.2

Reconsider the previous example, but change the output of H_1 to $y_1 = x_2 + e_1$. From the expression

$$\dot{V}_1 = -kx_2^2 + x_2 e_1 = -k(y_1 - e_1)^2 - e_1^2 + y_1 e_1$$

H_1 is passive, but we cannot conclude strict passivity or output strict passivity. Therefore, we cannot apply Theorem 7.2.

$$V = V_1 + V_2 = \frac{1}{4} a x_1^4 + \frac{1}{2} x_2^2 + \frac{1}{2} b x_3^2 + \frac{1}{2} x_4^2$$

as a Lyapunov function candidate for the closed-loop system

$$\begin{aligned}\dot{V} &= -kx_2^2 + x_2 e_1 - x_4^4 + x_4 e_2 \\ &= -kx_2^2 - x_2 x_4 - x_4^4 + x_4(x_2 - x_4) \\ &= -kx_2^2 - x_4^4 - x_4^2 \leq 0\end{aligned}$$

Example 7.2

Moreover, $\dot{V} = 0 \Rightarrow x_2 = x_4 = 0$ and

$$x_2(t) \equiv 0 \Rightarrow a x_1^3(t) - x_4(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

$$x_4(t) \equiv 0 \Rightarrow -b x_3(t) + x_2(t) - x_4(t) \equiv 0 \Rightarrow x_3(t) \equiv 0$$

Thus, by the invariance principle and the fact that V is radially unbounded, we conclude that the origin is globally asymptotically stable.

Strictly Passive + Passive memoryless

Theorem 7.3

Consider the feedback connection of a strictly passive dynamical system with a passive time-varying memoryless function. When $u = 0$, the origin of the closed-loop system is uniformly asymptotically stable. If the storage function for the dynamical system is radially unbounded, the origin will be globally uniformly asymptotically stable.

Proof: Let $V_1(x_1)$ be (positive definite) storage function of H_1.

$$\dot{V}_1 = \frac{\partial V_1}{\partial x_1} f_1(x_1, e_1) \leq e_1^T y_1 - \psi_1(x_1) = -e_2^T y_2 - \psi_1(x_1)$$

$$e_2^T y_2 \geq 0 \quad \Rightarrow \quad \dot{V}_1 \leq -\psi_1(x_1)$$

Time-invariant + Time-invariant memoryless

Theorem 7.4

Consider the feedback connection of a time-invariant dynamical system H_1 with a time-invariant memoryless function H_2. Suppose H_1 is zero-state observable and has a positive definite storage function $V_1(x)$, which satisfies

$$e_1^T y_1 \geq \dot{V}_1 + y_1^T \rho_1(y_1), \qquad e_2^T y_2 \geq e_2^T \varphi_2(e_2)$$

Then, the origin of the closed-loop system (when $u = 0$) is asymptotically stable if

$$v^T[\rho_1(v) + \varphi_2(v)] > 0, \qquad \forall\, v \neq 0$$

Furthermore, if V_1 is radially unbounded, the origin will be globally asymptotically stable.

Example 7.5

Consider the feedback connection of

$$H_1: \begin{cases} \dot{x}_1 = f(x) + G(x)e_1 \\ y_1 = h(x) \end{cases} \qquad\qquad H_2: y_2 = \sigma(e_2)$$

$$e_i, y_i \in R^m, \qquad \sigma(0) = 0, \qquad e_2^T \sigma(e_2) > 0, \forall e_2 \neq 0$$

Suppose H_1 is zero-state observable and there is a radially unbounded positive definite function $V_1(x)$ such that

$$\frac{\partial V_1}{\partial x} f(x) \leq 0, \qquad \frac{\partial V_1}{\partial x} G(x) = h^T(x), \qquad \forall x \in R^n$$

Both components are passive. Moreover, H_2 is satisfies

$$e_2^T y_2 = e_2^T \sigma(e_2)$$

Example 7.5

Apply Theorem 7.4

$$e_1^T y_1 \geq \dot{V}_1 + y_1^T \rho_1(y_1) \text{ is satisfied with } \rho_1 = 0,$$

$$e_2^T y_2 \geq e_2^T \varphi_2(e_2) \text{ is satisfied with } \varphi_2 = \sigma,$$

$$v^T[\rho_1(v) + \varphi_2(v)] = v^T \sigma(v) > 0, \quad \forall\, v \neq 0,$$

Hence, the origin of the closed-loop system is globally asymptotically stable.