Control Systems Stability and Robust Control

Relative Degree and Zero Dynamics

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Example

- Consider the controlled van der Pol equation:

$$\dot{x}_1 = x_2/\varepsilon, \qquad \dot{x}_2 = \varepsilon\left[-x_1 + x_2 - \frac{1}{3}x_2^3 + u\right], \qquad y = x_1$$

$$\dot{y} = \dot{x}_1 = x_2/\varepsilon, \qquad \ddot{y} = \dot{x}_2/\varepsilon = -x_1 + x_2 - \frac{1}{3}x_2^3 + u$$

  Hence, the system has relative degree two over R^2.

- Consider the same system with the different that

$$y = x_2$$

$$\dot{y} = \varepsilon\left[-x_1 + x_2 - \frac{1}{3}x_2^3 + u\right],$$

  Hence, the system has relative degree one over R^2.

Model and Lie Derivative

- Consider the n-dimensional, single-input-single-output system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

where $f, g,$ and h are sufficiently smooth in a domain D, $f: \ D \to R^n$ and $g: \ D \to R^n$ are called vector fields on D.

- The derivative of y is given by

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g(x)u] \stackrel{\text{def}}{=} L_f h(x) + L_g h(x)u$$

Model and Lie Derivative-Cont'd

- Lie Derivative of h with respect to f or along f.

$$L_f h(x) = \frac{\partial h}{\partial x} f(x)$$

- This is the derivative of h along the trajectories of the system $\dot{x} = f(x)$.
- The new notion is convenient when we repeat the calculation of the derivative with respect to the same vector field or a new one. For example,

$$L_g L_f h(x) = \frac{\partial \left(L_f h\right)}{\partial x} g(x)$$

$$L_f^2 h(x) = L_f L_f h(x) = \frac{\partial \left(L_f h\right)}{\partial x} f(x)$$

$$L_f^k h(x) = L_f L_f^{k-1} h(x) = \frac{\partial \left(L_f^{k-1} h\right)}{\partial x} f(x)$$

Relative Degree

- The first derivative

$$\dot{y} = L_f h(x) + L_g h(x) u$$

$$L_g h(x) = 0 \quad \Rightarrow \quad \dot{y} = L_f h(x)$$ (independent of u)

- The second derivative

$$y^{(2)} = \frac{\partial (L_f h)}{\partial x}[f(x) + g(x)u] = L_f^2 h(x) + L_g L_f h(x) u$$

$$L_g L_f h(x) = 0 \quad \Rightarrow \quad y^{(2)} = L_f^2 h(x)$$ (independent of u)

- The third derivative

$$y^{(3)} = L_f^3 h(x)$$ (independent of u)

Relative Degree-Cont'd

- The ρth derivative

$$L_g L_f^{i-1} h(x) = 0, \quad i = 1,2,\ldots,\rho-1; \qquad L_g L_f^{\rho-1} h(x) \neq 0$$

$$y^{(\rho)} = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x) u$$

Definition (Relative Degree)

The system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree ρ, $1 \leq \rho \leq n$, in $\mathcal{R} \subset D$ if $\forall x \in \mathcal{R}$.

$$L_g L_f^{i-1} h(x) = 0, \quad i = 1,2,\ldots,\rho-1; \qquad L_g L_f^{\rho-1} h(x) \neq 0$$

Example for Calculating a Relative Degree

- Consider the system

$$\dot{x}_1 = x_2/\varepsilon, \qquad \dot{x}_2 = \varepsilon\left[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u\right]$$

$$y = \frac{1}{2}(\varepsilon^2 x_1^2 + x_2^2)$$

- The derivative is given by

$$\dot{y} = \varepsilon^2 x_1 \dot{x}_1 + x_2 \dot{x}_2 = \varepsilon x_2^2 - (\varepsilon/3)x_2^4 + \varepsilon x_2 u$$

Hence, the system has relative degree one in $\{x_2 \neq 0\}$.

Exercise for Calculating a Relative Degree

- Find the relative degree for the following system

$$\dot{x}_1 = d_1(-x_1 - x_2x_3 + V_a) \quad d_1 > 0$$

$$\dot{x}_2 = d_2[-f_e(x_2) + u] \quad d_2 > 0$$

$$\dot{x}_3 = d_3(x_1x_2 - bx_3) \quad d_3 > 0$$

$$y = x_3$$

- The derivatives are given by

$$\dot{y} = \dot{x}_3 = d_3(x_1x_2 - bx_3)$$

$$\ddot{y} = d_3(x_1\dot{x}_2 + \dot{x}_1x_2 - b\dot{x}_3) = (\cdots) + d_2d_3x_1u$$

The system has relative degree two in $\{x_1 \neq 0\}$.

Relative Degree of Transfer Functions

- Consider a linear system represented by the transfer function

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0} \stackrel{\text{def}}{=} \frac{N(s)}{D(s)}, (m < n, b_m \neq 0)$$

Its relative degree is $m - n$. Does the definition coincide with the previous one?

- The state-space equations of the transfer function is given by

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & & & \cdots & 0 \\ 0 & 0 & 1 & \cdots & & & \cdots & 0 \\ \vdots & & \ddots & & & & & \vdots \\ & & & \ddots & & & & \\ & & & & \ddots & & & \vdots \\ \vdots & & & & & \ddots & & 0 \\ 0 & & & & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_m & \cdots & \cdots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \cdots & \cdots & b_m & 0 & \cdots & 0 \end{bmatrix}$$

Relative Degree of Transfer Functions-Cont'd

- $\dot{y} = CAx + CBu, \ddot{y} = CA^2x + CABu, \ldots, y^{(i)} = CA^ix + CA^{i-1}Bu$
- CA^i is obtained by shifting the elements of C two positions to the right.

  When $i = n - m - 1$, we have

$$CA^{n-m-1}B = b_m \neq 0$$

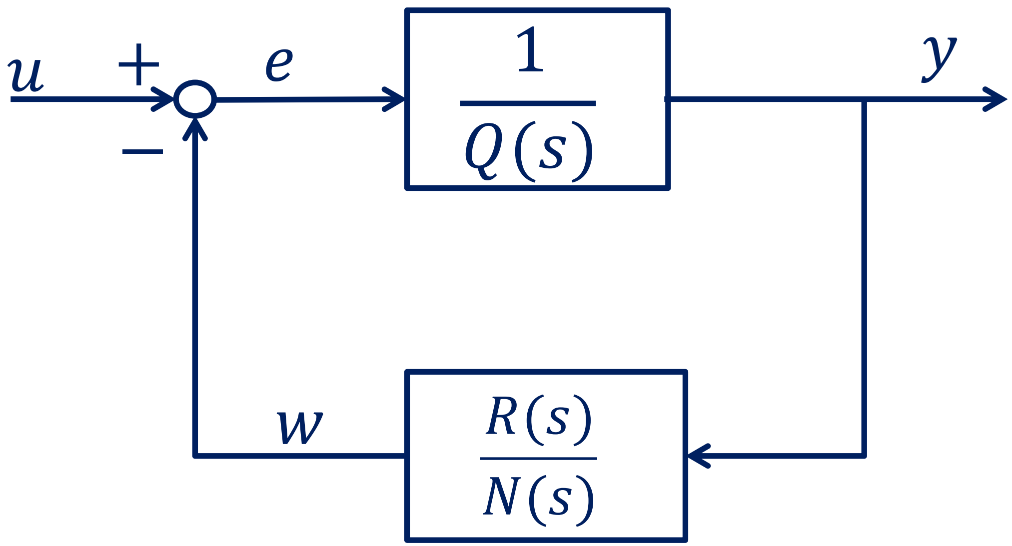
$$y^{(n-m)} = CA^{n-m}x + CA^{n-m-1}Bu \Rightarrow \rho = n - m$$

- These two definitions coincide!

Normal Form of Transfer Functions

- Let $D(s) = Q(s)N(s) + R(s)$, $Q(s), R(s)$ are the quotient and remainder polynomials, respectively. It follows that $\deg Q = n - m = \rho$, $\deg R < m$, the leading coefficient of $Q(s)$ is $1/b_m$.

$$H(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{Q(s)N(s) + R(s)} = \frac{\frac{1}{Q(s)}}{1 + \frac{1}{Q(s)}\frac{R(s)}{N(s)}}$$



Normal Form of Transfer Functions-Cont'd

- State model of $1/Q(s)$: $\xi = \text{col}\left(y, \dot{y}, \ldots, y^{(\rho-1)}\right)$

$$\dot{\xi} = (A_c + B_c\lambda^T)\xi + B_c b_m e, \quad y = C_c\xi, \quad \lambda \in R^\rho$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ 0 & \ldots & \ldots & 0 & 0 \end{bmatrix}, \; B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \; C_c = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

Normal Form of Transfer Functions-Cont'd

- State model of $R(s)/N(s)$

$$\dot{\eta} = A_0\eta + B_0 y, \qquad w = C_0\eta$$

- Normal Form

$$\dot{\eta} = A_0\eta + B_0 C_c \xi \qquad \text{(internal model)}$$

$$\dot{\xi} = A_c\xi + B_c(\lambda^T\xi - b_m C_0\eta + b_m u) \quad \text{(external model)}$$

$$y = C_c\xi$$

The eigenvalues of A_0 are the zeros of $H(s)$.

- Can this idea be generalized to the following nonlinear system?

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

Example

- Consider the controlled van der Pol equation:

$$\dot{x}_1 = x_2/\varepsilon, \quad \dot{x}_2 = \varepsilon\left[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u\right], \quad y = x_2$$

$$\dot{y} = \dot{x}_2 = \varepsilon\left[-x_1 + x_2 - \frac{1}{3}x_2^3 + u\right] \Rightarrow \quad \rho = 1$$

- The system is in the normal form with $\eta = x_1$ and $\xi = x_2$

$$y(t) \equiv 0 \quad \Rightarrow \quad x_2(t) \equiv 0 \quad \Rightarrow \quad \dot{x}_1 = 0$$

- How to choose the variables η and ξ?

Normal Form of Nonlinear Systems

- Consider the system with relative degree ρ

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

- Find a change of variables $z = [\eta^T, \xi^T]^T$ such that the new state z can be partitioned into a ρ-dimensional vector ξ and $(n - \rho)$-dimensional vector η, where η does not depends on u and y does not depends on η.

Normal Form of Nonlinear Systems-Cont'd

- The internal variables are defined as

$$z = T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \phi(x) \\ --- \\ \psi(x) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix}$$

- Choose $\phi(x)$ such that $T(x)$ is a diffeomorphism and

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad \text{for } 1 \le i \le n-\rho, \quad \forall x \in D_x$$

Normal Form of Nonlinear Systems-Cont'd

- The normal form is then given by

$$\dot{\eta} = \frac{\partial \phi}{\partial x}[f(x) + g(x)u] = f_0(\eta, \xi)$$

$$\dot{\xi}_i = \xi_{i+1}, \qquad 1 \le i \le \rho - 1$$

$$\dot{\xi}_\rho = L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)u$$

$$y = \xi_1$$

- The equation for ξ can be written as $\dot{\xi} = A_c \xi + B_c \left[L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)u\right]$ and $y = C_c \xi$
- Does the transformation always exists?

Normal Form of Nonlinear Systems-Cont'd

Theorem

Suppose the system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree $\rho(\leq n)$ in $\mathcal{R}$. If $\rho = n$, then for every $x_0 \in \mathcal{R}$, a neighborhood N of x_0 exists such that the map $T(x) = \psi(x)$, restricted to N, is a diffeomorphism on N. If $\rho < n$, then, for every $x_0 \in \mathcal{R}$, a neighborhood N of x_0 and smooth functions $\phi_1(x), \dots, \phi_{n-\rho}(x)$ exist such that

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \qquad \text{for } 1 \leq i \leq n - \rho$$

is satisfied for all $x \in N$ and the map $T(x) = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}$, restricted to N, is a diffeomorphism on N.

Zero Dynamics and Minimum Phase

- Consider the normal form

$$\dot{\eta} = f_0(\eta, \xi)$$
$$\dot{\xi} = A_c\xi + B_c\left[L_f^{\rho}h(x) + L_gL_f^{\rho-1}h(x)u\right]$$
$$y = C_c\xi$$

- When the output is identically zero, we have

$$y(t) \equiv 0 \Rightarrow \xi(t) \equiv 0 \Rightarrow \dot{\eta} = f_0(\eta, 0)$$

Definition

The equation $\dot{\eta} = f_0(\eta, 0)$ is called the zero dynamics of the system. The system is said to be minimum phase if the zero dynamics have an asymptotically stable equilibrium point in the domain of interest.

Exercise

- Consider the controlled van der Pol equation:

$$\dot{x}_1 = x_2/\varepsilon, \quad \dot{x}_2 = \varepsilon\left[-x_1 + x_2 - \frac{1}{3}x_2^3 + u\right], \quad y = x_2$$

$$\dot{y} = \dot{x}_2 = \varepsilon\left[-x_1 + x_2 - \frac{1}{3}x_2^3 + u\right] \Rightarrow \quad \rho = 1$$

- Find the zero dynamics of the proposed system and see if it is minimum-phase?

- The system is in the normal form with $\eta = x_1$ and $\xi = x_2$

$$y(t) \equiv 0 \quad \Rightarrow \quad x_2(t) \equiv 0 \quad \Rightarrow \quad \dot{x}_1 = 0$$

No asymptotically stable equilibrium point, hence, non-minimum phase.