Nonlinear Control

Stabilization via Linearization & Feedback Linearization

Prof. Fei Chen

Northeastern University

State Feedback Stabilization

- Consider the control system $\dot{x} = x + u$. We can construct $u = -kx$, where $k > 1$. The resulting closed-loop system is given by $\dot{x} = -(k-1)x$. It can be shown that $x = 0$ is globally asymptotically stable. Why?
- State feedback stabilization

$$\dot{x} = f(x, u) \qquad [f(0,0) = 0]$$

Find u such that the origin is an asymptotically.

- If $u = \phi(x)$, then the control is called "static feedback".
- If $u = \phi(x, z)$ with $\dot{z} = g(x, z)$, the control is called "dynamic feedback".

Local/Region/Semiglobal/Global Stabilization-Example

- Consider the system

$$\dot{x} = x^2 + u$$

- Linearization:

$$\dot{x} = u, \qquad u = -kx, \qquad k > 0$$

Linearized closed-loop system is $\dot{x} = -kx$. Thus, $u = -kx$ achieves local stabilization.

- Closed-loop system:

$$\dot{x} = -kx + x^2$$

The region of attraction is $\{|x| < k\}$. Why? Thus, control $u = -kx$ achieves regional stabilization.

Local/Region/Semiglobal/Global Stabilization-Example

- But it achieves semiglobal stabilization because any compact set $\{|x| \leq r\}$ can be included in the region of attraction by choosing $k > r$.
- The control

$$u = -x^2 - kx$$

achieves global stabilization because it yields the linear closed-loop system $\dot{x} = -kx$ whose origin is globally exponentially stable.

Local/Region/Semiglobal/Global Stabilization-Definition

$$\dot{x} = f(x, u), \qquad u = \phi(x)$$

- Local Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is asymptotically stable
- Regional Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is asymptotically stable and a given region G is a subset of the region of attraction (for all $x(0) \in G$, $\lim_{t\to\infty} x(t) = 0$)
- Semiglobal Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is asymptotically stable and $\phi(x)$ can be designed such that any given compact set (no matter how large) can be included in the region of attraction.
- Global Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is globally asymptotically stable.

Stabilization of Linear Systems

- Consider the system

$$\dot{x} = Ax + Bu$$

where (A, B) is stabilizable. Find $u = -Kx$ such that the system is stabilized.

- The closed-loop system is given by

$$\dot{x} = (A - BK)x$$

- Find K such that $(A - BK)$ is Hurwitz. If (A, B) is stabilizable, then we can always find such a K.

Stabilization via Linearization- Example

- Consider the pendulum equation

$$\ddot{\theta} = -sin\theta - b\dot{\theta} + cu$$

We want to stabilize the pendulum at $\theta = \delta_1$.

- The stationary control input is $0 = -sin\delta_1 + cu_{ss}$. Define the new variable as

$$x_1 = \theta - \delta_1, \qquad x_2 = \dot{\theta}, \qquad u_\delta = u - u_{ss}$$

The system can be rewritten in terms of the new variables as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -[\sin(x_1 + \delta_1) - sin\delta_1] - bx_2 + cu_\delta$$

- Linearization yields

$$A = \begin{bmatrix} 0 & 1 \\ -\cos(x_1 + \delta_1) & -b \end{bmatrix}_{x_1=0} = \begin{bmatrix} 0 & 1 \\ -\cos\delta_1 & -b \end{bmatrix}; B = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

Stabilization via Linearization- Example

- (A,B) is controllable. Why?
- Take $K = [k_1 \quad k_2]$, leading to

$$A - BK = \begin{bmatrix} 0 & 1 \\ -(\cos\delta_1 + ck_1) & -(b + ck_2) \end{bmatrix}$$

By choosing $k_1 > -\frac{\cos\delta_1}{c},\ k_2 > -\frac{b}{c}$, $A - BK$ is Hurwitz.

- The control input is given by

$$u = \frac{\sin\delta_1}{c} - Kx = \frac{\sin\delta_1}{c} - k_1(\theta - \delta_1) - k_2\dot{\theta}$$

- The stabilization results hold locally!

Stabilization via Linearization

- Consider the system:

$$\dot{x} = f(x, u)$$

where $f(0,0) = 0$ and f is continuously differentiable in a domain $D_x \times D_u$ that contains the origin $(x = 0, u = 0)$ $(D_x \subset R^n, D_u \subset R^m)$

- Linearization of the system yields

$$\dot{x} = Ax + Bu$$

$$A = \frac{\partial f}{\partial x}(x,u)\bigg|_{x=0,u=0}; \qquad \mathrm{B} = \frac{\partial f}{\partial u}(x,u)\bigg|_{x=0,u=0}$$

- If (A, B) is stabilizable, we can find a matrix K such that $(A - BK)$ is Hurwitz. The system can be locally stabilized by $u = Kx$.

Stabilization via Linearization- Exercise

- Find a local stabilizer of for the following system

$$\dot{x} = -x^3 + u$$

- The linearized system is given by

$$\dot{x} = u$$

The control input can be designed as

$$u = -kx, \qquad k > 0.$$

Stabilization via Feedback Linearization- Example

- Consider the following system

$$\dot{x}_1 = asinx_2, \qquad \dot{x}_2 = -x_1^2 + u$$

where $D = \{|x_2| < \pi/2\}$.

- Introduce the transformation

$$z = T(x) = \begin{bmatrix} x_1 \\ asinx_2 \end{bmatrix} \Rightarrow \dot{z} = \begin{bmatrix} z_2 \\ \sqrt{a^2 - z_2^2}(-z_1^2 + u\ ) \end{bmatrix}$$

where the domain is given by $T(D) = \{|z_2| < a\}$

- Let $\mathrm{u} = (v + z_1^2)(a^2 - z_2^2)^{-\frac{1}{2}}$, then system can be transformed as

$$z = Az + Bu$$

where $A = [0\ 1\ ; 0\ 0], B = [0; 1]$.

Stabilization via Feedback Linearization- Example

- Let $v = Kz$, where

$$K = [\sigma^2 \quad 2\sigma], \sigma > 0, \Longrightarrow \lambda(A - BK) = -\sigma, -\sigma$$

- The system is stabilized by

$$u = (v + z_1^2)(a^2 - z_2^2)^{-\frac{1}{2}} = (\sigma^2 z_1 + 2\sigma z_2 + z_1^2)(\mathrm{a}^2 - \mathrm{z}_2^2)^{-\frac{1}{2}}$$

$$= (\sigma^2 x_1 + 2\sigma a\ sin x_2 + x_1^2)(\mathrm{a}^2 - (a sin x_2)^2)^{-\frac{1}{2}}$$

$$= (\sigma^2 x_1 + 2\sigma a\ sin x_2 + x_1^2)(\mathrm{a}\ \mathrm{cos x}_2)^{-1}$$

Stabilization via Feedback Linearization

- Consider the nonlinear system

$$\dot{x} = f(x) + G(x)u$$

$$f(0) = 0, \qquad x \in R^n, \qquad u \in R^m$$

- Suppose there is a change of variables $z = T(x)$, defined for all $x \in \mathrm{D} \subset R^n$, that transforms the system into the controller form

$$\dot{z} = Az + B[\psi(x) + \gamma(x)u]$$

where (A, B) is controllable and $\gamma(x)$ is nonsingular for all $x \in \mathrm{D}$

$$u = \gamma^{-1}(x)[-\psi(x) + v] \Longrightarrow \dot{z} = Az + Bv$$

Stabilization via Feedback Linearization

- Design the control input

$$v = -Kz$$

  such that $(A - BK)$ is Hurwitz.

- The origin $z = 0$ of the closed-loop system

$$\dot{z} = (A - BK)z$$

  is globally exponentially stable.

- The actual control input is

$$u = \gamma^{-1}(x)[-\psi(x) - KT(x)]$$

- The closed-loop system in the x-coordinates:

$$\dot{x} = f(x) + G(x)\gamma^{-1}(x)[-\psi(x) - KT(x)] \stackrel{\text{def}}{=} f_c(x)$$

The Effect of Model Uncertainty

- Stabilization via feedback linearization requires

$$u = \gamma^{-1}(x)[-\psi(x) - KT(x)]$$

What is the effect of uncertainty in ψ, γ and T?

- Let $\hat{\psi}(x), \hat{\gamma}(x)$ and $\hat{T}(x)$ be nominal models of $\psi(x), \gamma(x)$ and $T(x)$

$$u = \hat{\gamma}^{-1}(x)\left[-\hat{\psi}(x) - K\hat{T}(x)\right]$$

- The closed-loop system is then given by

$$\dot{z} = Az + B\left\{\psi(x) + \gamma(x)\hat{\gamma}^{-1}(x)\left[-\hat{\psi}(x) - K\hat{T}(x)\right]\right\}$$

or equivalently

$$\dot{z} = (A - BK)z + B\Delta(\mathrm{z})$$

where $\Delta(z) = \left\{\psi(x) + \gamma(x)\hat{\gamma}^{-1}(x)\left[-\hat{\psi}(x) - K\hat{T}(x)\right] + KT(x)\right\}$.

Robustness to Model Uncertainties

Theorem

Consider the system

$$\dot{z} = (A - BK)z + B\Delta(\mathrm{z})$$

Let P be the solution to the Lyapunov equation

$$P(A - BK) + (A - BK)^T P = -I$$

- If $||\Delta(z)|| \leq k||z||$ for $\forall z \in D_{\mathrm{z}}, k < 1/(2||PB||)$, then the origin is exponentially stable. It is globally exponentially stable if $D_{\mathrm{z}} = R^n$.
- If $||\Delta(z)|| \leq k||z|| + \delta$ for $\forall z \in D_{\mathrm{z}}$ and $B_r \subset D_{\mathrm{z}}$, then there exist positive constants c_1 and c_2 such that if $\delta < c_1 r$ and $z(0) \in \{z^T P z \leq \lambda_{min}(P) r^2\}$, $||z(t)||$ will be ultimately bounded by δc_2 .if $D_{\mathrm{z}} = R^n$, $||z(t)||$will be globally ultimately bounded by δc_2 for any $\delta > 0$.

Example

- Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin(x_1 + \delta_1) - bx_2 + cu$$

Define the following input

$$u = \left(\frac{1}{c}\right)[\sin(x_1 + \delta_1) - (k_1 x_1 + k_2 x_2)]$$

The closed-loop system is given by

$$A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -(k_2 + b) \end{bmatrix}$$

- Now suppose c is uncertain, then we have to use the estimate $\hat{c}$, leading to

$$u = \left(\frac{1}{\hat{c}}\right)[\sin(x_1 + \delta_1) - (k_1 x_1 + k_2 x_2)]$$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -k_1 x_1 - (k_2 + b)x_2 + \Delta(x)$$

$$\Delta(x) = \left(\frac{c - \hat{c}}{\hat{c}}\right)[\sin(x_1 + \delta_1) - (k_1 x_1 + k_2 x_2)]$$

Example

- The uncertainty is upper bounded by

$$|\Delta(x)| \leq k\|x\| + \delta\,, \qquad \forall x$$

$$k = \left|\frac{c-\hat{c}}{\hat{c}}\right|\left(1+\sqrt{k_1^2+k_2^2}\right), \qquad \delta = \left|\frac{c-\hat{c}}{\hat{c}}\right| |\sin\delta_1|$$

- Let P be the solution to the Lyapunov equation

$$P(A-BK)+(A-BK)^T P = -I, \qquad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

- We have

$$k < \frac{1}{2\sqrt{p_{12}^2+p_{22}^2}} \Longrightarrow \mathrm{GUB}$$

- Additionally, if

$$\sin\delta_1 = 0 \;\Longrightarrow \mathrm{GES}$$