Nonlinear Control

Lecture # 14

Control Lyapunov Functions

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Control Lyapunov Functions

- Consider the stabilization problem for the system

$$\dot{x} = f(x) + g(x)u, \qquad f(0) = 0, x \in R^n, u \in R$$

- Suppose there is a locally Lipschitz stabilizing state feedback control $u = \chi(x)$ such that the origin of

$$\dot{x} = f(x) + g(x)\chi(x)$$

is asymptotically stable.
- Then, by the converse Lyapunov theorem, there is a smooth Lyapunov function $\mathrm{V}(x)$ such that

$$\frac{\partial V}{\partial x}[f(x) + g(x)\chi(x)] < 0, \qquad \forall x \neq 0$$

in some neighborhood of the origin.
- If $u = \chi(x)$ is globally stabilizing, then $\mathrm{V}(x)$ is radially unbounded

$$\frac{\partial V}{\partial x} g(x) = 0 \text{ and } x \neq 0 \Rightarrow \frac{\partial V}{\partial x} f(x) < 0$$

Control Lyapunov Functions

Definition

A positive definite function $V(x)$ is a Control Lyapunov Function (CLF) for the system $\dot{x} = f(x) + g(x)u$ if

$$\frac{\partial V}{\partial x} g(x) = 0 \quad and \quad x \neq 0 \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x) < 0 \qquad (*)$$

It is a Global Control Lyapunov Function if it is radially unbounded.

The system $\dot{x} = f(x) + g(x)u$ is stabilizable by a state feedback control only if it has a CLF. It turns out that the existence of CLF is also sufficient!

Control Lyapunov Functions

Sontag's Formula:

$$\emptyset(x) = \begin{cases} -\dfrac{\dfrac{\partial V}{\partial x} f + \sqrt{\left(\dfrac{\partial V}{\partial x} f\right)^2 + \left(\dfrac{\partial V}{\partial x} g\right)^4}}{\left(\dfrac{\partial V}{\partial x} g\right)}, & if \ \dfrac{\partial V}{\partial x} g \neq 0 \\ 0, & if \ \dfrac{\partial V}{\partial x} g = 0 \end{cases} \tag{9.35}$$

Control Lyapunov Functions

- This can be seen by using V as a Lyapunov function candidate for the closed-loop system

$$\dot{x} = f(x) + g(x)\emptyset(x)$$

- For $x \neq 0$, if $\frac{\partial V}{\partial x} g = 0$, then

$$\dot{V} = \frac{\partial V}{\partial x} f < 0$$

- If $\frac{\partial V}{\partial x} g \neq 0$, then

$$\dot{V} = \frac{\partial V}{\partial x} f - \left[\frac{\partial V}{\partial x} f + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4} \right]$$

$$= -\sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4} < 0$$

which shows that the origin is asymptotically stable. If V is a global control Lyapunov function, the origin is globally asymptotically stable.

Control Lyapunov Functions->Stabilizing Control Input

- Consider the following system

$$\dot{x} = -x + u$$

- A control Lyapunov function can be chosen as $V(x) = \frac{1}{2}x^2$
- Under the Sontag's formula, the control input can be given by

$$u = \phi(x) = x - \sqrt{2}x$$

Control Lyapunov Functions

Lemma 9.6

If $f(x), g(x)$ and $V(x)$ are smooth then $\phi(x)$, defined by (9.35), will be smooth for $x \neq 0$. If they are of class $C^{\ell+1}$ for $\ell \geq 1$, then $\phi(x)$ will be of class C^{ℓ}. Continuity at $x = 0$ follows from one of the following two cases.

- $\phi(x)$ is continuous at $x = 0$ if $V(x)$ has the small control property; namely, given any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \neq 0$ and $\|x\| < \delta$, then there is u with $\|u\| < \varepsilon$ such that

$$\frac{\partial V}{\partial x}[f(x) + g(x)u] < 0$$

- $\phi(x)$ is locally Lipschitz at $x = 0$ if there is a locally Lipschitz function $\chi(x)$, with $\chi(0) = 0$, such that

$$\frac{\partial V}{\partial x}[f(x) + g(x)\chi(x)] < 0, \qquad \text{for } x \neq 0$$

Control Lyapunov Functions

How can we find a CLF?

- If we know of any stabilizing control with a corresponding Lyapunov function V, then V is a CLF
- Feedback Linearization

$$\dot{x} = f(x) + G(x)u, \qquad z = T(x), \qquad \dot{z} = (A - BK)z$$

$$P(A - BK) + (A - BK)^T P = -Q, \qquad Q = Q^T > 0$$

$$\mathrm{V} = z^T P z = T^T(x) P T(x) \text{ is a CLF}$$

- Backstepping

Example

- The scalar system

$$\dot{x} = x - x^3 + u$$

- Feedback linearization:

$$u = \chi(x) = -x + x^3 - \alpha x \quad (\alpha > 0) \;\Rightarrow\; \dot{x} = -\alpha x,$$

The closed-loop system $\dot{x} = -\alpha x$ has the Lyapunov function $\mathrm{V}(x) = \frac{1}{2}x^2$.

$$\frac{\partial V}{\partial x} g = x, \qquad \frac{\partial V}{\partial x} f = x(x - x^3)$$

$$\emptyset(x) = -\frac{\frac{\partial V}{\partial x} f + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4}}{\left(\frac{\partial V}{\partial x} g\right)}$$

$$= -\frac{x(x - x^3) + \sqrt{(x^2(x - x^3)^2)^2 + x^4}}{x}$$

$$\Rightarrow \qquad \emptyset(x) = -x + x^3 - x\sqrt{(1 - x^2)^2 + 1}$$

Compare with

$$\chi(x) = -x + x^3 - \alpha x \qquad \Rightarrow \qquad \alpha = \sqrt{2}$$

Example 9.17

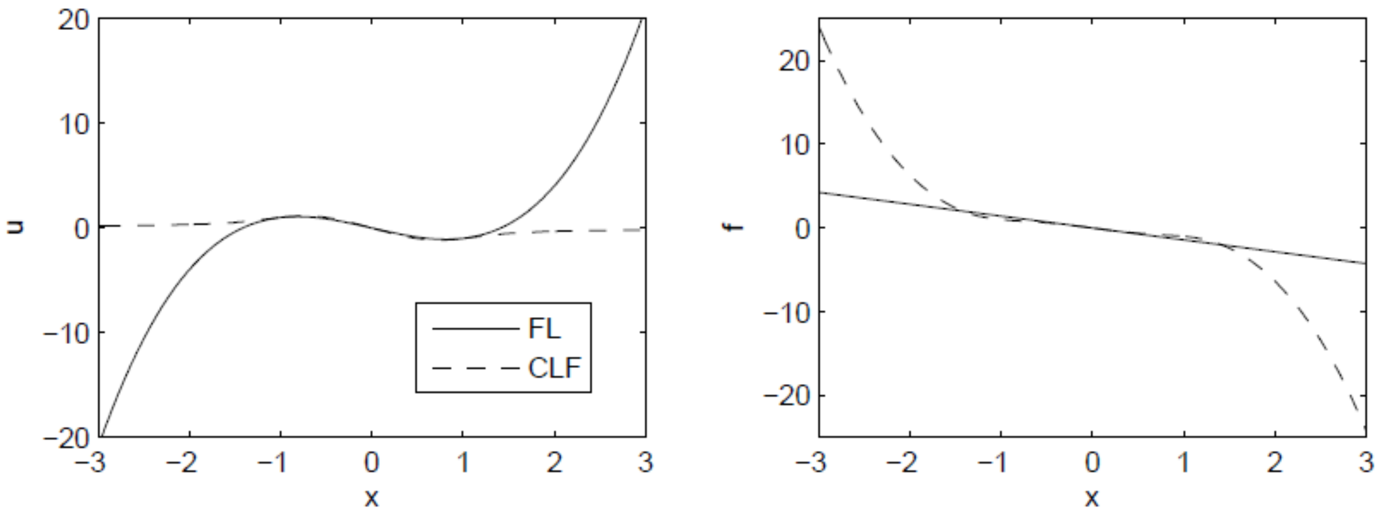

Figure 9.1: Comparison of the control u and the closed-loop $\dot{x} = f$ between feedback linearization (FL) and control Lyapunov function (CLF) in Example 9.17.

Robustness Property

- Consider the perturbed system $\dot{x} = f(x) + g(x)ku$. If the control u stabilizes the system for all $k \in [\alpha, \beta]$, we say the system has a gain margin $[\alpha, \beta]$.

Lemma 9.7

Suppose f, g, and V satisfy the conditions of Lemma 9.6, V is a control Lyapunov function, and Ø is given by Sontag's formula (9.35). Then, the origin of

$$\dot{x} = f(x) + g(x)k\emptyset(x)$$

is asymptotically stable for all $k \geq \frac{1}{2}$. If V is a global control Lyapunov function, then the origin is globally asymptotically stable.

Exercise

- Design the stabilizing input for the systems via a control Lyapunov function

$$\dot{x} = -x^2 + u$$