Nonlinear Control

Sliding-Model Control

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What Is Sliding-Mode Control- Example

- Consider the two-dimensional, single-input system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = h(x) + g(x)u$$

where h and g are unknown locally Lipschitz function and $g(x) > g_0 > 0$ for all x.

- Design

$$s = ax_1 + x_2 = 0$$

$$s(t) \equiv 0 \quad \Rightarrow \quad \dot{x} = -ax_1$$

$$a > 0 \Rightarrow \lim_{t\to\infty} x_1(t) = 0$$

- How can we bring the trajectory to the manifold $s = 0$?

What Is Sliding-Mode Control- Example

- The dynamics of s is given by

$$\dot{s} = a\dot{x}_1 + \dot{x}_2 = ax_2 + h(x) + g(x)u$$

- Suppose

$$\left|\frac{ax_2 + h(x)}{g(x)}\right| \leq \varrho(x)$$

- With $\mathrm{V} = \frac{1}{2}s^2$ as a Lyapunov function candidate, we have

$$\dot{V} = s\dot{s} = s[ax_2 + h(x)] + g(x)su \leq g(x)|s|\varrho(x) + g(x)su$$

- Define $\mathrm{u} = -\beta(x)\mathrm{sgn}(s)$, where

$$\mathrm{sgn}(s) = \begin{cases} 1, & s > 0 \\ -1, & \mathrm{s} < 0 \end{cases}, \qquad \beta(x) \geq \varrho(x) + \beta_0, \qquad \beta_0 > 0$$

What Is Sliding-Mode Control- Example

- It follows that

$$\dot{V} \leq g(x)|s|\varrho(x) - g(x)\beta(x)|s| \leq -g(x)\beta_0|s|$$

We have

$$\dot{V} \leq -g_0\beta_0\sqrt{2V}$$

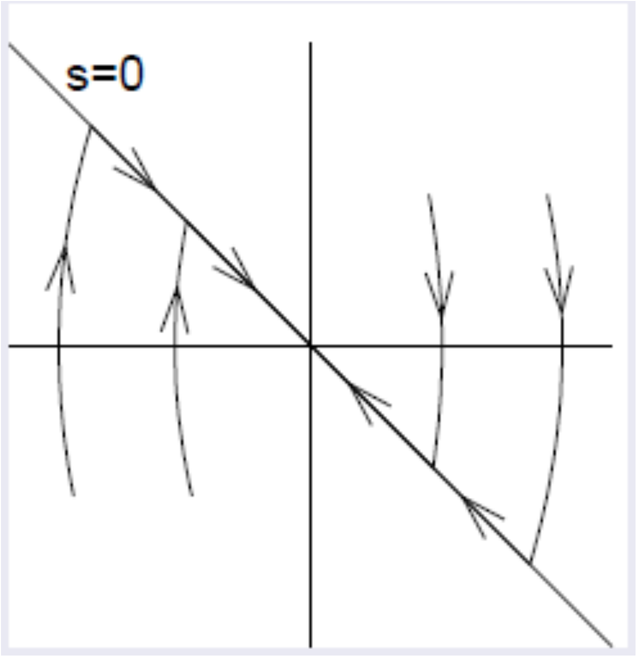
which further leads to

$$\frac{\mathrm{d}V}{\sqrt{V}} \leq -g_0\beta_0\sqrt{2}dt \Rightarrow \sqrt{V\big(s(t)\big)} \leq \sqrt{V\big(s(0)\big)} - g_0\beta_0\frac{1}{\sqrt{2}}t$$

- This indicates that $s = 0$ will be reached in finite time and will be maintained thereafter.

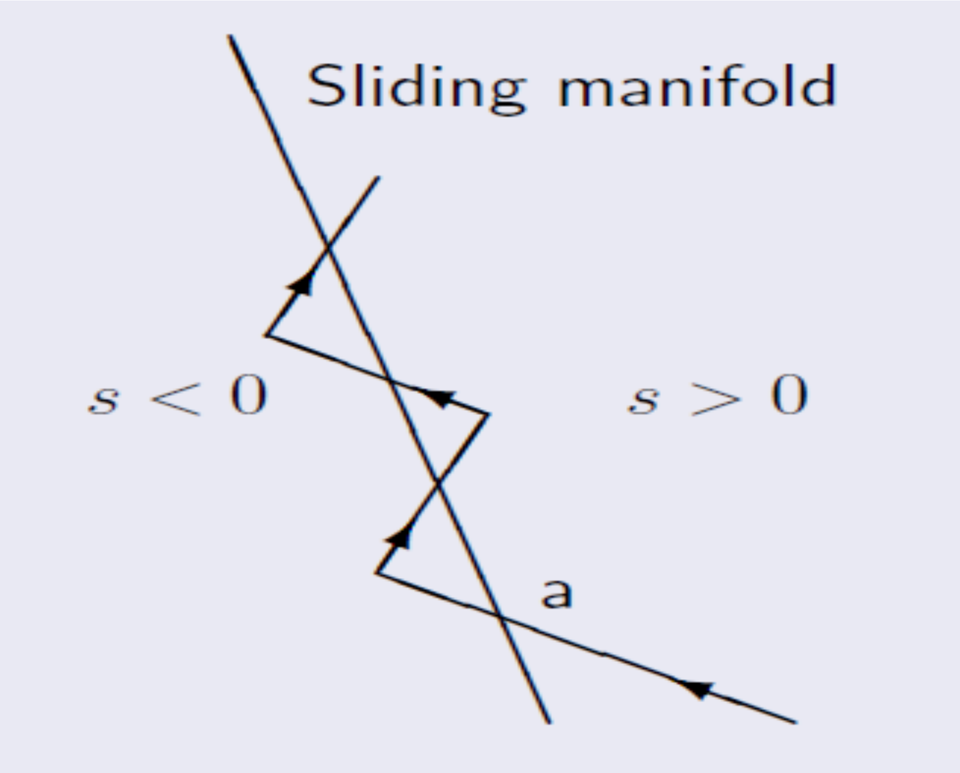
What Is Sliding-Mode Control?

- Reaching phase: reach the sliding-mode surface in finite time and maintained thereafter
- Sliding phase: on the sliding-mode surface, the system converges to the origin.



- What is the benefit of stabilization via sliding-mode control?

Chattering Effect



- How can we reduce the chattering effect?

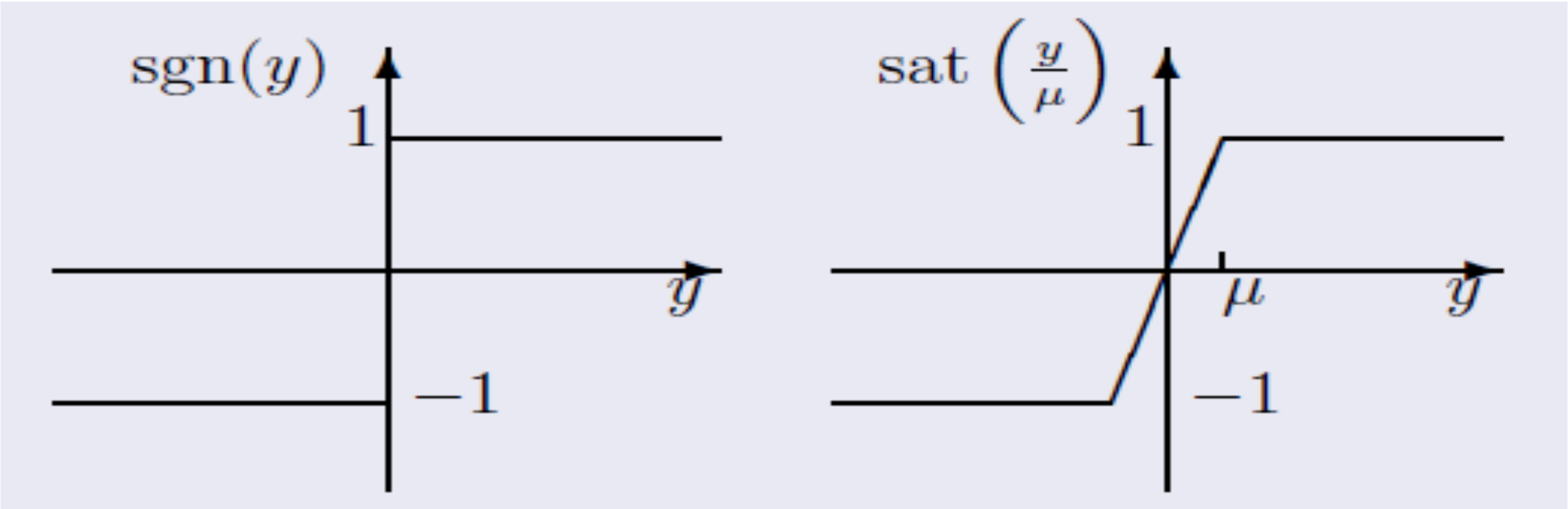
Pendulum Equation

- Replace the signum function by a high-slope saturation function

$$u = -\beta(x)\,\mathrm{sat}\left(\frac{s}{\mu}\right)$$

where the saturation function is given by

$$\mathrm{sat}(y) = \begin{cases} y, & if\ |y| \leq 1 \\ \mathrm{sgn}(y), & if\ |y| > 1 \end{cases}$$



Pendulum Equation

How can we analyze the system?

$$\text{For } |s| \geq \mu, \qquad u = -\beta(x)\,\mathrm{sgn}(s)$$

With $c \geq \mu$

- $\Omega = \left\{|x_1| \leq \frac{c}{a}, |s| \leq c\right\}$ is positively invariant
- The trajectory reaches the boundary layer $\{|s| \leq \mu\}$ in finite time
- The boundary layer is positively invariant

Pendulum Equation

Inside the boundary layer $|x| \leq \mu$

$$\dot{x}_1 = -ax_1 + s \qquad \dot{s} = ax_2 + h(x) - g(x)\beta(x)\frac{s}{\mu}$$

$$x_1\dot{x}_1 \leq -ax_1^2 + |x_1|\mu$$

$$x_1\dot{x}_1 \leq -(1-\theta_1)ax_1^2, \qquad \forall |x_1| \geq \frac{\mu}{\theta_1 a}, 0 < \theta_1 < 1$$

The trajectories reach the positively invariant set

$$\Omega_\mu = \left\{|x_1| \leq \frac{\mu}{\theta_1 a}, |s| \leq \mu\right\}$$

in finite time.