Control Systems Stability and Robust Control

Lyapunov Redesign and High-Gain Feedback

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What Is Lyapunov Redesign?

- Consider the perturbed system:

$$\dot{x} = f(x) + G(x)[u + \delta(t, x, u)], x \in R^n, u \in R^m$$

- The nominal model is given by

$$\dot{x} = f(x) + G(x)u$$

with the stabilizing law $u = \emptyset(x)$ satisfying

$$\frac{\partial V}{\partial x} [f(x) + G(x)\emptyset(x)] \leq -W(x), \forall x \in D, W \text{ is PD}$$

- Suppose $u = \emptyset(x) + v$, and the uncertainty satisfies

$$||\delta(t, x, \emptyset(x) + v)|| \leq \rho(x) + k_0||v||, 0 \leq k_0 < 1$$

What is Lyapunov Redesign?

- Under $u = \emptyset(x) + v$, the closed-loop system is given by

$$\dot{x} = f(x) + G(x)\emptyset(x) + G(x)[v + \delta(t, x, \emptyset(x) + v)]$$

- The derivative of the Lyapunov function is then given by

$$\dot{V} = \frac{\partial V}{\partial x}(f + G\emptyset) + \frac{\partial V}{\partial x}G(v + \delta)$$

- Let $w^T = \frac{\partial V}{\partial x}G$, leading to

$$\dot{V} \leq -W(x) + w^T v + w^T \delta$$

- The objective is to design v to such that the uncertainty term $w^T\delta$ can be covered by $w^T v$.
- The design of v to stabilize the perturbed system is called Lyapunov Redesign.

How to achieve Lyapunov Redesign?

- We have

$$w^T v + w^T \delta \leq w^T v + \|w\| \|\delta\| \leq w^T v + \|w\| [\rho(x) + k_0 \|v\|]$$

- Design

$$v = -\beta(x) \frac{w}{\|w\|} \qquad \left(\frac{w}{\|w\|} = \text{sgn}(w) \text{ for } m = 1 \right)$$

- We have

$$\begin{aligned} w^T v + w^T \delta &\leq -\beta \|w\| + \rho \|w\| + k_0 \beta \|w\| \\ &= -\beta (1 - k_0) \|w\| + \rho \|w\| \end{aligned}$$

which gives

$$\beta(x) \geq \frac{\rho(x)}{(1 - k_0)} \Rightarrow w^T v + w^T \delta \leq 0 \Rightarrow \dot{V} \leq -W(x)$$

Example 10.5 (Pendulum equation)

- Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -sin(x_1 + \delta_1) - b_0 x_2 + cu$$

$$0 \le b_0 \le 0.2, \qquad 0.5 \le c \le 2$$

- The system is feedback linearizable

$$\dot{x} = Ax + B[-\sin(x_1 + \delta_1) - b_0 x_2 + cu]$$

- Let the nominal parameters be $\hat{b}_0 = 0 \, and \, \hat{c}$. The nominal control input is given by

$$\emptyset(x) = (\frac{1}{\hat{c}}) [\sin(x_1 + \delta_1) - k_1 x_1 - k_2 x_2]$$

where $K = [k_1 \quad k_2]$ is chosen such that $A - BK$ is Hurwitz.

Example 10.5 (Pendulum equation)

- The actual control input is given by

$$u = \emptyset(x) + v$$

- The uncertainty is given by

$$\delta = \left(\frac{c - \hat{c}}{\hat{c}^2}\right)[sin(x_1 + \delta_1) - k_1 x_1 - k_2 x_2] - \frac{b_0}{\hat{c}} x_2 + \left(\frac{c - \hat{c}}{\hat{c}}\right) v$$

which is upper bounded by

$$|\delta| \leq \rho_0 + \rho_1|x_1| + \rho_2|x_2| + k_0|v|$$

with

$$\rho_0 \geq \left|\frac{(c - \hat{c})sin\delta_1}{\hat{c}^2}\right|, \ \rho_1 \geq \left|\frac{c - \hat{c}}{\hat{c}^2}\right|(1 + k_1)$$

$$\rho_2 \geq \frac{b_0}{\hat{c}} + |\frac{c - \hat{c}}{\hat{c}^2}|k_2, \quad k_0 \geq \left|\frac{c - \hat{c}}{\hat{c}}\right|$$

- Assume $k_0 < 1, \beta(x) \geq \beta_0 + \frac{\rho_0 + \rho_1|x_1| + \rho_2|x_2|}{1 - k_0}, \beta_0 > 0$

Continuous Implementation

- Design the continuous control input :

$$v = -\beta(x) Sat\left(\frac{\beta(x)w}{\mu}\right) = \begin{cases} -\beta(x)\dfrac{w}{\|w\|}, & \text{if } \beta(x)\|w\| \geq \mu \\ -\beta^2(x)\dfrac{w}{\mu}, & \text{if } \beta(x)\|w\| < \mu \end{cases}$$

- For $\beta(x)\|w\| \geq \mu$

$$\beta(x)\|w\| \geq \mu \Rightarrow \dot{V} \leq -W(x)$$

- For $\beta(x)\|w\| < \mu$

$$\begin{aligned} \dot{V} &\leq -W(x) + w^T\left[-\beta^2 . \frac{w}{\mu} + \delta\right]. \\ &\leq -W(x) - \frac{\beta^2}{\mu}\|w\|^2 + \rho\|w\| + k_0\|w\|\|v\| \\ &= -W(x) - \frac{\beta^2}{\mu}\|w\|^2 + \rho\|w\| + \frac{k_0\beta^2}{\mu}\|w\|^2 \end{aligned}$$

Continuous Implementation

$$\dot{V} \leq -W(x) + (1 - k_0)\left(-\frac{\beta^2}{\mu}\|w\|^2 + \beta\|w\|\right)$$

- Using $-\frac{y^2}{\mu} + y \leq \frac{\mu}{4}$, for $y \geq 0$, yields

$$\dot{V} \leq -W(x) + \mu\frac{(1 - k_0)}{4}$$

- Define the following Class K functions

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad W(x) \geq \alpha_3(\|x\|)$$

- For $0 < \theta < 1$, we have

$$\dot{V} \leq -(1 - \theta)\alpha_3(\|x\|)) - \theta\alpha_3(\|x\|) + \mu(1 - k_0)/4.$$

$$\leq -(1 - \theta)\alpha_3(\|x\|), \forall\|x\| \geq {\alpha_3}^{-1}\left(\frac{\mu(1 - k_0)}{4\theta}\right) \stackrel{\text{def}}{=} \mu_0$$

Theorem 10.3

Under the foregoing assumptions, for any $x(t_0) \in \{V(x) \leq \alpha_1(r)\}$, the solution of the closed-loop system satisfies

$$||x(t)|| \leq \max\{\beta_1(||x(t_0)||, t - t_0), b(\mu)\}$$

where β_1 is a class KL function and

$$b(\mu) = \alpha_1{}^{-1}(\alpha_2(\mu_0) = \alpha_1{}^{-1}(\alpha_2(\alpha_3{}^{-1}(0.25\mu(1 - k_0)/\theta)))$$

If all the assumptions hold globally and $\alpha_1 \in K_\infty$, then the above inequality holds for any initial state $x(t_0)$.

Theorem 10.4

If

$$W(x) \geq \varphi(x)^2(x), \qquad \beta(x) \geq \beta_0 > 0, \qquad \rho(x) \leq \rho_1 \varphi(x)$$

For sufficiently small μ, the origin is uniformly asymptotically stable. If

$$c_1 ||x||^2 \leq V(x) \leq c_2 ||x||^2, \qquad W(x) \geq c_3 ||x||^2$$

For $c_i > 0$, then the origin is exponentially stable.

Revisit Sliding Mode Control

- Consider

$$\dot{x} = f(x) + B(x)[G(x)u + \delta(t, x, u)]$$

$x \in R^m, u \in R^m, f$ and B are known, while G and δ could be uncertain, $f(0) = 0$, $G(x)$ is a positive definite symmetric matrix with

$$\lambda_{\min}\big(G(x)\big) \geq \lambda_0 > 0$$

- Regular form: by introducing $\begin{bmatrix}\eta \\ \xi\end{bmatrix} = T(x), \frac{\partial T}{\partial x} B(x) = \begin{bmatrix}0 \\ I\end{bmatrix}$, the system can be written as

$$\dot{\eta} = f_a(\eta, \xi)$$

$$\dot{\xi} = f_b(\eta, \xi) + G(x)u + \delta(t, x, u)$$

Revisit Sliding Mode Control-Cont'd

- Let $\phi(\eta)$ be a stabilizing law of the η system
- Define the sliding-model manifold

$$s = \xi - \phi(\eta) = 0, \phi(0) = 0$$

$$s(t) \equiv 0 \Rightarrow \dot{\eta} = f_a(\eta, \phi(\eta))$$

- Design the input u such that the system $s = 0$ is achieved. The dynamics of s is given by

$$\dot{s} = f_b(\eta, \xi) - \frac{\partial \phi}{\partial \eta} f_a(\eta, \xi) + G(x)u + \delta(t, x, u)$$

The control input takes the following form $u = \psi(\eta, \xi) + v$.

Revisit Sliding Mode Control-Cont'd

- Typical choices of ψ

$$\psi = 0 \text{ or } \psi = -\hat{G}^{-1}\left[f_b - \frac{\partial \phi}{\partial \eta} f_a\right]$$

- The dynamics of s can be rewritten as

$$\dot{s} = G(x)\nu + \Delta(t, x, u)$$

- Assume that

$$\left\|\frac{\Delta(t, x, v)}{\lambda_{min}\big(G(x)\big)}\right\| \leq \rho(x) + k_0\|v\|, \forall (t, x, v) \in [0, \infty)\times D\times R^m$$

$$\rho(x) \geq 0, \quad 0 \leq k_0 < 1$$

Revisit Sliding Mode Control-Cont'd

- Define $V = \frac{1}{2} s^T s$. It follows that

$$\dot{V} = s^T \dot{s} = s^T G(x) v + s^T \Delta(x, t, v)$$

- The control input can be given as

$$v = -\beta(x) \frac{s}{||s||}, \qquad \beta(x) \geq \frac{\rho(x)}{1 - k_0} + \beta_0, \beta_0 > 0$$

- Trajectories reach the manifold $s = 0$ in finite time and cannot leave it.

Revisit Sliding Mode Control-Cont'd

- In order to avoid the chattering effect, design

$$Sat(y) = \begin{cases} y, \|y\| \le 1 \\ \dfrac{y}{\|y\|}, \|y\| > 1 \end{cases}$$

$$v = -\beta(x) Sat\left(\frac{s}{\mu}\right)$$

$$\|s\| \ge \mu \Rightarrow Sat\left(\frac{s}{\mu}\right) = \frac{s}{\|s\|} \Rightarrow s^T \dot{s} \le -\lambda_0 \beta_0 (1 - k_0)\|s\|$$

- Trajectories reach the boundary layer $\{\|s\| \le \mu\}$ in finite time and remains ins ide thereafter.

High-Gain Feedback

- Replace $v = \beta(x)\, Sat\,(\frac{s}{\mu})$ by the high-gain feedback $v = \dfrac{\beta(x)s}{\mu}$
- Let

$$V = \frac{1}{2}\, s^T s$$

- The derivative of V is given by

$$\begin{aligned}
\dot{V} &= -\frac{\beta}{\mu} s^T G s + s^T \Delta \\
&\leq -\frac{\beta}{\mu} \lambda_{\min}(G) \|s\|^2 + \lambda_{\min}(G)\, \rho \|s\| + \lambda_{\min}(G) k_0 \frac{\beta}{\mu} \|s\|^2. \\
&\leq \lambda_{\min}(G) \left[-\left(\frac{\|s\|}{\mu} - 1 \right) \beta(1 - k_0)\|s\| - \beta_0(1 - k_0)\|s\| \right] \\
&\leq -\lambda_0 \beta_0 (1 - k_0)\|s\|, \qquad \text{for } \|s\| \geq \mu
\end{aligned}$$

- The trajectories reach $\{\|s\| \leq \mu\}$ in finite time

Comparison of Sliding-Mode Control and High-Gain Feedback

- What is the difference from sliding mode control?
- The trajectories reach the boundary layer faster because

$$\mu \dot{s} = -G(x)s + \mu \Delta$$

- The trade-off is the large spike in the control signal.
- Theorem 10.3 and 10.4 hold for high-gain feedback.

Example

Recall Example 10.1 where the pendulum is stabilized at

$$\theta = \frac{\pi}{2} \text{ and } s = \theta - \frac{\pi}{2} + \dot{\theta}$$

Sliding Mode: $u = -(2.5 + 2|\dot{\theta}|)\, sat(s/\mu)$

High-gain Feedback: $u = -(2.5 + 2|\dot{\theta}|)(s/\mu)$