Nonlinear Control

Global Observers and High-Gain Observers

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Global Observers

- Consider the following system

$$\dot{x} = Ax + \psi(u, y), \qquad y = Cx$$

- Design the observer

$$\dot{\hat{x}} = A\hat{x} + \psi(u, y) + H(y - C\hat{x})$$

$$\tilde{x} = x - \hat{x}$$

$$\dot{\tilde{x}} = (A - HC)\tilde{x}$$

- Design H such that $A - HC$ is Hurwitz

$$\lim_{t \to \infty} \tilde{x}(t) = 0, \qquad \forall \tilde{x}(0)$$

Global Observers-A More General Case

- Consider the system

$$\dot{x} = Ax + \psi(u, y) + \phi(x, u), \quad y = Cx$$

$$\|\phi(x, u) - \phi(z, u)\| \leq L\|x - z\|$$

- Design the observer

$$\dot{\hat{x}} = A\hat{x} + \psi(u, y) + \phi(\hat{x}, u) + H(y - C\hat{x})$$

$$\dot{\tilde{x}} = (A - HC)\tilde{x} + \phi(x, u) - \phi(\hat{x}, u)$$

- Design the Lyapunov function

$$P(A - HC) + (A - HC)^T P = -I, \qquad V = \tilde{x}^T P \tilde{x}$$

$$\dot{V} = -\tilde{x}^T \tilde{x} + 2\tilde{x}^T P[\phi(x, u) - \phi(\hat{x}, u)] \leq -\|\tilde{x}\|^2 + 2L\|P\|\|\tilde{x}\|^2$$

$$L < \frac{1}{2\|P\|} \quad \Rightarrow \lim_{t \to \infty} \tilde{x}(t) = 0, \qquad \forall \tilde{x}(0)$$

High-Gain Observers-Example

- Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \phi(x,u), \qquad y = x_1$$

- Design the observer

$$\dot{\hat{x}}_1 = \hat{x}_2 + h_1(y - \hat{x}_1), \qquad \dot{\hat{x}}_2 = \phi_0(\hat{x},u) + h_2(y - \hat{x}_1)$$

$$|\phi_0(\mathrm{z},u) - \phi(x,u)| \leq L\|x - z\| + M$$

- The error system is given by

$$\dot{\tilde{x}} = A_0\tilde{x} + B\delta(x,\tilde{x},u)$$

$$A_0 = \begin{bmatrix} -h_1 & 1 \\ -h_2 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \delta = \phi(x,u) - \phi_0(\hat{x},u)$$

- Design h_1 and h_2 such that A_0 is Hurwitz.

High-Gain Observers-Example

- Transfer function from δ to $\tilde{x}$:

$$G_0(s) = \frac{1}{s^2 + h_1 s + h_2}\begin{bmatrix} 1 \\ s + h_1 \end{bmatrix}$$

- We can make $\sup_{w \in R} \|G_0(jw)\|$ arbitrarily small by choosing

$$h_2 \gg h_1 \gg 1$$

- By choosing $h_1 = \frac{\alpha_1}{\varepsilon}$, $h_2 = \frac{\alpha_2}{\varepsilon^2}$, $\varepsilon \ll 1$

$$G_0(s) = \frac{\varepsilon}{(\varepsilon s)^2 - \alpha_1 \varepsilon s + \alpha_2}\begin{bmatrix} \varepsilon \\ \varepsilon s + \alpha_1 \end{bmatrix}$$

$$\lim_{\varepsilon \to 0} G_0(s) = 0$$

High-Gain Observers-Example

- Let

$$\eta_1 = \frac{\tilde{x}_1}{\varepsilon} \qquad \eta_2 = \tilde{x}_2$$

- It follows that

$$\varepsilon\dot{\eta} = F\eta + \varepsilon B\delta, \text{ where } F = \begin{bmatrix} -\alpha_1 & 1 \\ -\alpha_2 & 0 \end{bmatrix}$$

$$|\delta| \leq L\|\tilde{x}\| + M \leq L\|\eta\| + M$$

- Design the Lyapunov function

$$PF + F^T P = -I, \qquad V = \eta^T p\eta$$

$$\begin{aligned} \varepsilon\dot{V} &= -\eta^T\eta + 2\varepsilon\eta PB\delta \\ &\leq -\|\eta\|^2 + 2\varepsilon L\|PB\|\|\eta\|^2 + 2\varepsilon M\|PB\|\|\eta\| \end{aligned}$$

High-Gain Observers   Example

$$\varepsilon L\|PB\| \le \frac{1}{4} \;\Rightarrow\; \varepsilon\dot{V} \le -\frac{1}{2}\|\eta\|^2 + 2\varepsilon M\|PB\|\|\eta\|$$

$$\|\eta(t)\| \le \max\{\mathrm{k}\mathrm{e}^{-at/\varepsilon}\|\eta(0)\|, \varepsilon cM\}, \qquad \forall t \ge 0$$

- Peaking Phenomenon

$$x_1(0) \ne \hat{x}_1(0) \;\Rightarrow\; \eta_1(0) = O(1/\varepsilon)$$

- The solution will contain a term of the form $(1/\varepsilon)\ \mathrm{e}^{-at/\varepsilon}$

Example (revisited)

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - 2x_2 + a x_1^2 x_2 + b \sin 2t \,, y = x_1$$

a = 0.25, b = 0.2, and Ω =$\{1.5x_1^2 + x_1 x_2 + 0.5x_2^2 \leq \sqrt{2}\}$ is positively invariant

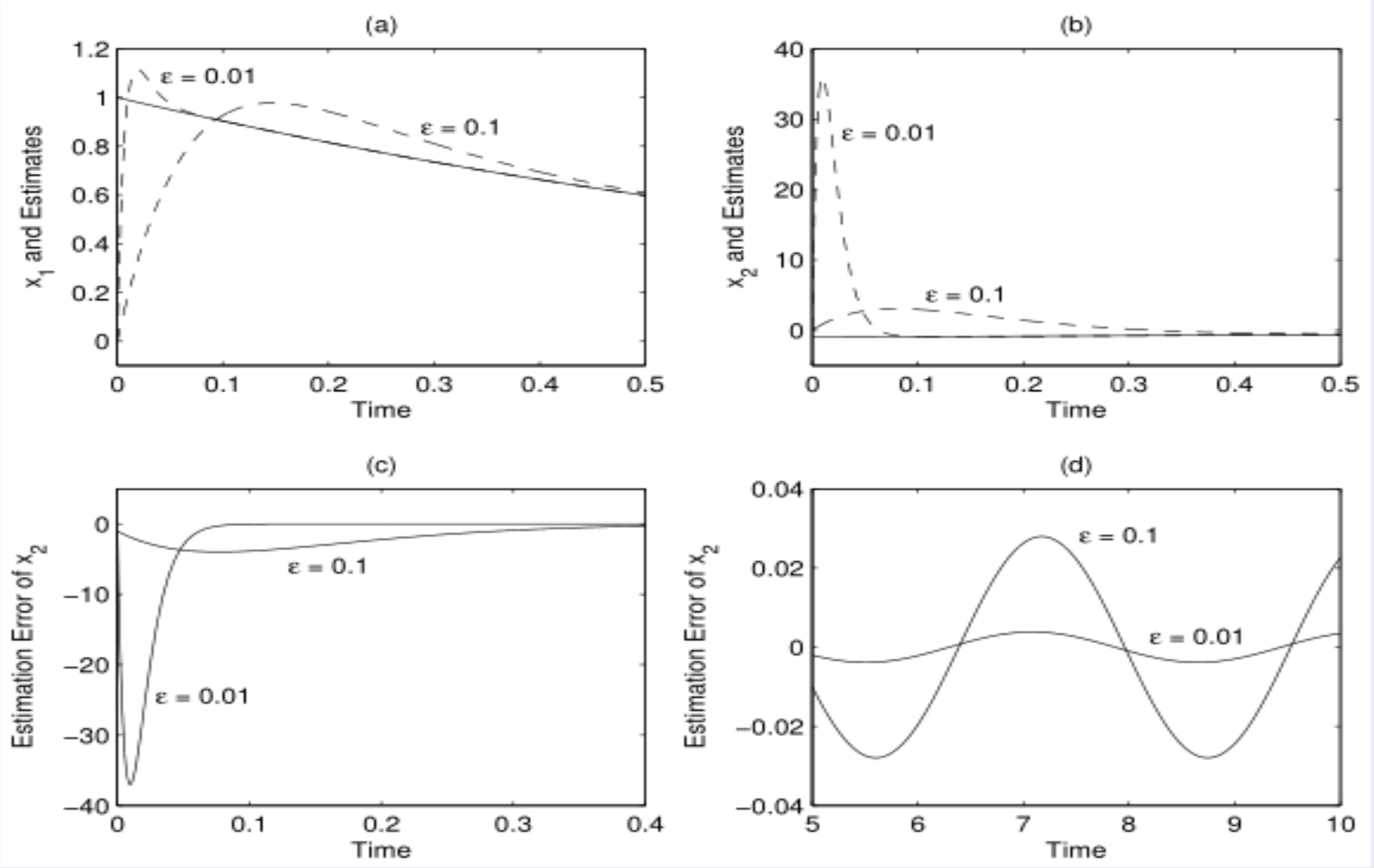
$$\dot{\hat{x}}_1 = \hat{x}_2 + \frac{2}{\varepsilon}(y - \hat{x}_1)$$

$$\dot{\hat{x}}_2 = -\hat{x}_1 - 2\hat{x}_2 + \hat{a}\hat{x}_1^2\,\hat{x}_2 + \hat{b} sin2t + \frac{1}{\varepsilon^2}(y - \hat{x}_1)$$

Case 1:$\hat{a} = 0.25$ and $\hat{b} = 0.2\big(Figures(a) and\ (b)\big)$

Case 2:$\hat{a} = \hat{b} = 0\big(Figures(c) and\ (d)\big)$

Example (revisited)



Measurement Noise

$$y = x_1 + v, \qquad |v(t)| \le N$$

$$\varepsilon\dot{\eta} = F\eta + \varepsilon B\delta - \frac{1}{\varepsilon}Ev, \text{ where } E = \begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix}$$

$$\varepsilon\dot{V} \le -\frac{1}{2}\|\eta\|^2 + 2\varepsilon M\|PB\|\|\eta\| + \frac{2N}{\varepsilon}\|PE\|\|\eta\|$$

- Ultimate bound:

$$\|\tilde{x}\| \le c_1 M\varepsilon + \frac{c_2 N}{\varepsilon}$$

Measurement Noise