Control Systems Stability and Robust Control

Tracking & Regulation

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Tracking & Regulation Problem

- Normal form

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi}_i &= \xi_{i+1}, \quad \text{for } 1 \le i \le \rho - 1 \\ \dot{\xi}_\rho &= a(\eta, \xi) + b(\eta, \xi)u \\ y &= \xi_1. \end{aligned}$$

where $\eta \in D_\eta \subset R^{n-\rho}, \xi = \mathrm{col}(\xi_1, \dots, \xi_\rho) \in D_\xi \subset R^\rho$.

- Tracking Problem : Design a feedback controller such that

$$\lim_{t \to \infty} [y(t) - r(t)] = 0$$

while ensuring boundedness of all state variables.

- Regulation Problem: r is constant.

Assumption

Assumption 13.1

$$b(\eta, \xi) \geq b_0 > 0, \forall \eta \in D_\eta, \xi \in D_\xi$$

Assumption 13.2

$\dot{\eta} = f_0(\eta, \xi)$ is bounded-input–bounded-state stable over $D_\eta \times D_\xi$

Assumption 13.3

$r(t)$ and its derivatives up to $r^{(\rho)}(t)$ are bounded for all $t \geq 0$ and the ρth derivative $r^{(\rho)}(t)$ is a piecewise continuous function of t. Moreover, $\mathcal{R} = \text{col}\left(r, \dot{r}, \ldots, r^{(\rho-1)}\right) \in D_\xi$ for all $t \geq 0$

Change of variables

- Define the tracking errors

$$e_1 = \xi_1 - r, e_2 = \xi_2 - r^{(1)}, \dots, e_\rho = \xi_\rho - r^{(\rho-1)}$$

- The system can be rewritten in terms of $\mathrm{e_i}$ as

$$\dot{\eta} = f_0(\eta, \xi)$$

$$\dot{e}_i = e_{i+1}, \textit{for}\ 1 \le i \le \rho - 1$$

$$\dot{e}_\rho = a(\eta, \xi) + b(\eta, \xi)u - r^{(\rho)}$$

- Goal : Ensure $\mathrm{e} = \mathrm{col}(\mathrm{e_1}, \dots, \mathrm{e_\rho}) = \xi - \mathcal{R}$ is bounded for all $\mathrm{t} \ge 0$ and converges to zero as t tends to infinity.

Tracking

- The system can be written in a compact form as

$$\dot{\eta} = f_0(\eta, \xi),$$

$$\dot{e} = A_c e + B_c[a(n, \xi) + b(\eta, \xi)u - r^\rho]$$

- Feedback linearization :

$$u = [-a(n, \xi) + r^\rho + v]/b(\eta, \xi)$$

$$\dot{\eta} = f_0(\eta, \xi),$$

$$\dot{e} = A_c e + B_c v$$

$$v = -Ke, A_c - B_c K \text{ is Hurwitz}$$

$$\dot{\eta} = f_0(\eta, \xi), \dot{e} = (A_c - B_c K)e$$

- $A_c - B_c K$ Hurwitz $\Rightarrow$ $\mathrm{e}(t)$ is bounded and $\lim_{t\to\infty} e(t) = 0$

Example (Pendulum equation)

- Consider the pendulum equation

$$\dot{x}_1 = x_2, \dot{x}_2 = -\sin x_1 - bx_2 + cu, \qquad y = x_1$$

- We want the output y to track a reference signal $r(t)$

$$e_1 = x_1 - r, e_2 = x_2 - \dot{r}$$

$$\dot{e}_1 = e_2, \dot{e}_2 = -\sin x_1 - bx_2 + cu - \ddot{r}$$

$$u = \frac{1}{c}[\sin x_1 + bx_2 + \ddot{r} - k_1 e_1 - k_2 e_2]$$

where $K = [k_1, k_2]$ assigns the eigenvalues of $A_c - B_c K$ at desired loc

ations in the open left-half complex plane.

Simulation

$$r = \sin\left(\frac{t}{3}\right), x(0) = \mathrm{col}\left(\frac{\pi}{2}, 0\right)$$

Nominal: $b = 0.03, c = 1$ Figures (a) and (b)

Perturbed: $b = 0.015, c = 0.5$ Figure (c)

Reference (dashed)

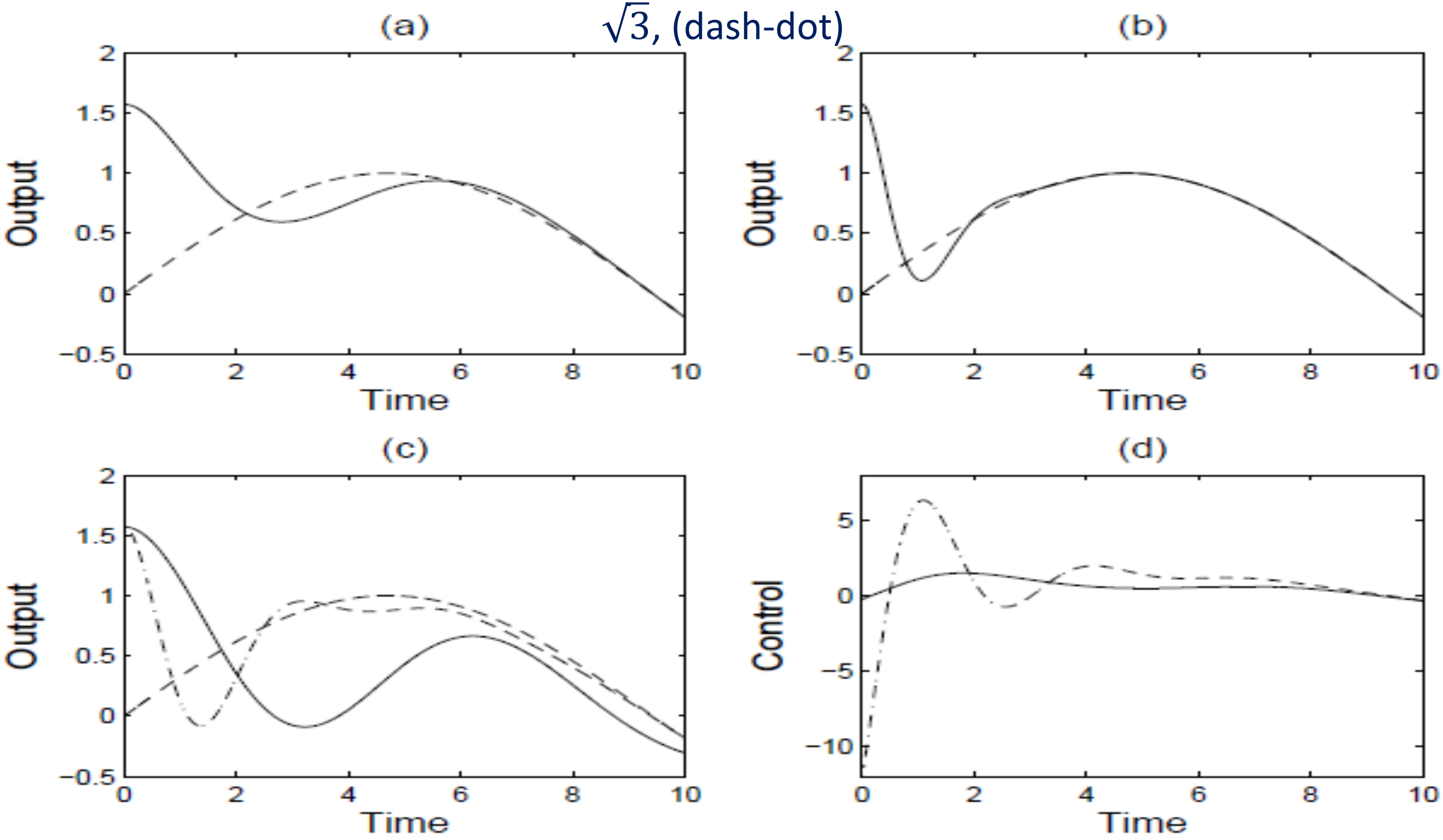
Low gain: $K = [1\ 1], \lambda = -0.5 \pm j0.5\sqrt{3}$, (solid)

High gain: $K = [9\ 3], \lambda = -1.5 \pm j1.5\sqrt{3}$, (dash-dot)

Simulation

Nominal: $b = 0.03, c = 1$ Figures (a) and (b); Perturbed: $b = 0.015, c = 0.5$ Figure (c)

Low gain: $K = [1\ 1], \lambda = -0.5 \pm j0.5\sqrt{3}$, (solid); High gain: $K = [9\ 3], \lambda = -1.5 \pm j1.5\sqrt{3}$, (dash-dot)



Robust Tracking(Sliding mode control)

- Consider the perturbed system:

$$\dot{\eta} = f_0(\eta, \xi)$$

$$\dot{e}_i = e_{i+1}, 1 \le i \le \rho - 1$$

$$\dot{e}_\rho = a(n, \xi) + b(\eta, \xi)u + \delta(t, \eta, \xi, u) - r^\rho(t)$$

- Design the sliding surface

$$s = \left(k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1}\right) + e_\rho = 0$$

$$\dot{s} = \sum_{i=1}^{\rho-1} k_i e_{i+1} + a(n, \xi) + b(\eta, \xi)u + \delta(t, \eta, \xi, u) - r^\rho(t)$$

- The control input is designed as

$u = v$ or $u = -\frac{1}{\hat{b}(\eta,\xi)}\left[\sum_{i=1}^{\rho-1} k_i e_{i+1} + \hat{a}(n, \xi) - r^\rho(t)\right] + v$

Robust Tracking(Sliding mode control)

- In either case, we have

$$\dot{s} = b(\eta, \xi)v + \Delta(t, \eta, \xi, v)$$

- Suppose that

$$\left|\frac{\Delta(t, \eta, \xi, v)}{b(\eta, \xi)}\right| \leq \varrho(\eta, \xi) + \kappa_0 |v|, 0 \leq \kappa_0 \leq 1$$

- Design the sliding-mode control input as

$$v = -\beta(\eta, \xi)\text{sat}\left(\frac{s}{\mu}\right), \beta(\eta, \xi) \geq \frac{\varrho(\eta, \xi)}{1 - \kappa_0} + \beta_0, \beta_0 > 0$$

- It follows that $s\dot{s} \leq -\beta_0 b_0 (1 - \kappa_0)|s|, |s| \geq \mu$

Example

- Design tracking control law for the following system

$$\dot{e}_1 = e_2, \dot{e}_2 = -\sin x_1 - bx_2 + cu - \ddot{r}$$

$$r = \sin\left(\frac{t}{3}\right), 0 \le b \le 0.1, 0.5 \le c \le 2$$

- Design

$$s = e_1 + e_2$$

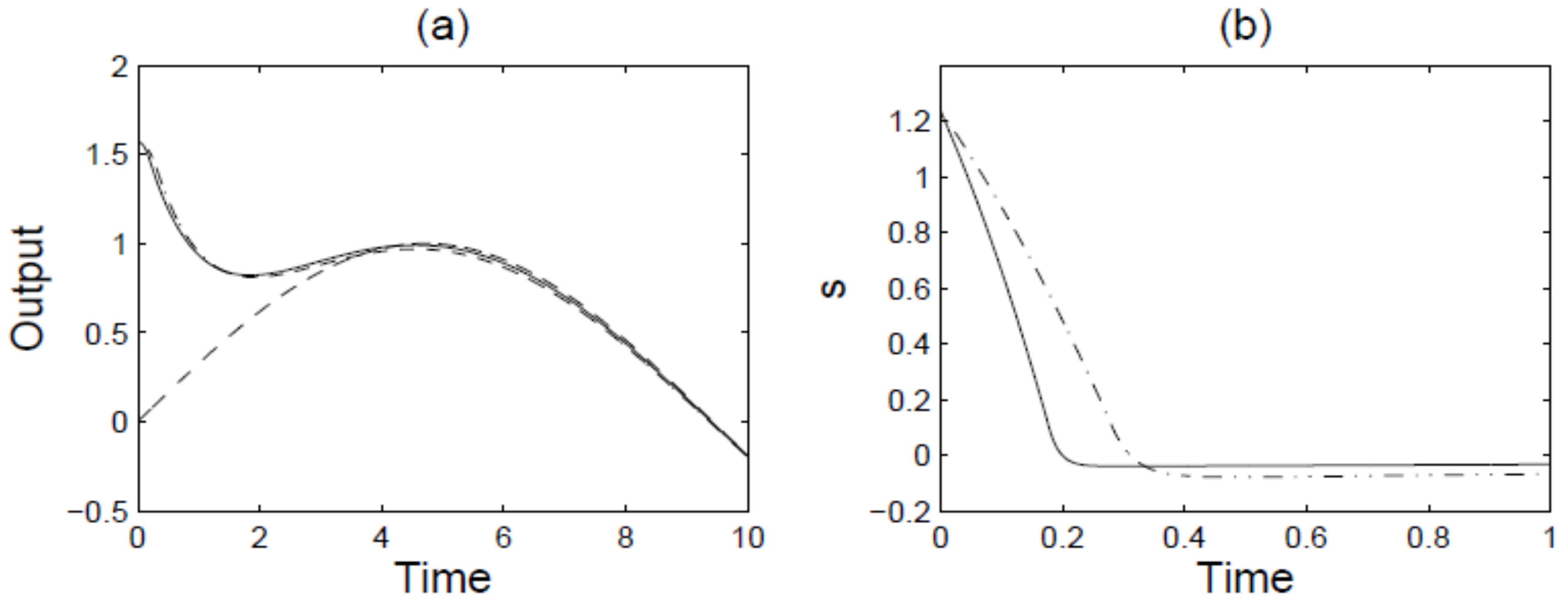
$$\dot{s} = e_2 - \sin x_1 - bx_2 + cu - \ddot{r}$$

$$= (1-b)e_2 - \sin x_1 - b\dot{r} - \ddot{r} + cu$$

$$\left|\frac{(1-b)e_2 - \sin x_1 - b\dot{r} - \ddot{r}}{c}\right| \le \frac{|e_2| + 1 + 0.1/3 + 1/9}{0.5}$$

$$u = -(2|e_2| + 3)\mathrm{sat}\left(\frac{e_1 + e_2}{\mu}\right)$$

Simulation



- $b = 0.03, c = 1(\text{solid})$
- $b = 0.015, c = 0.5(\text{dash} - \text{dot})$
- Reference (dashed)