Control Systems Stability and Robust Control

Lecture #2

Stability Concepts & Linearization

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Equilibrium points

$$\dot{x} = f(x)$$

- f is locally Lipschitz over a domain $D \subset R^n$
- Suppose $\bar{x} \in D$ is an equilibrium point; that is, $f(\bar{x}) = 0$.
- For convenience, we assume $\bar{x} = 0$. No loss of generality, since by defining $y = x - \bar{x}$

$$\dot{y} = \dot{x} = f(x) = f(y + \bar{x}) \stackrel{\text{def}}{=} g(y)$$

where $g(0) = 0$.

Stability Concepts

The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

- stable if for each $\varepsilon > 0$ there is $\delta > 0$(dependent on ε) such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \forall t \geq 0$$

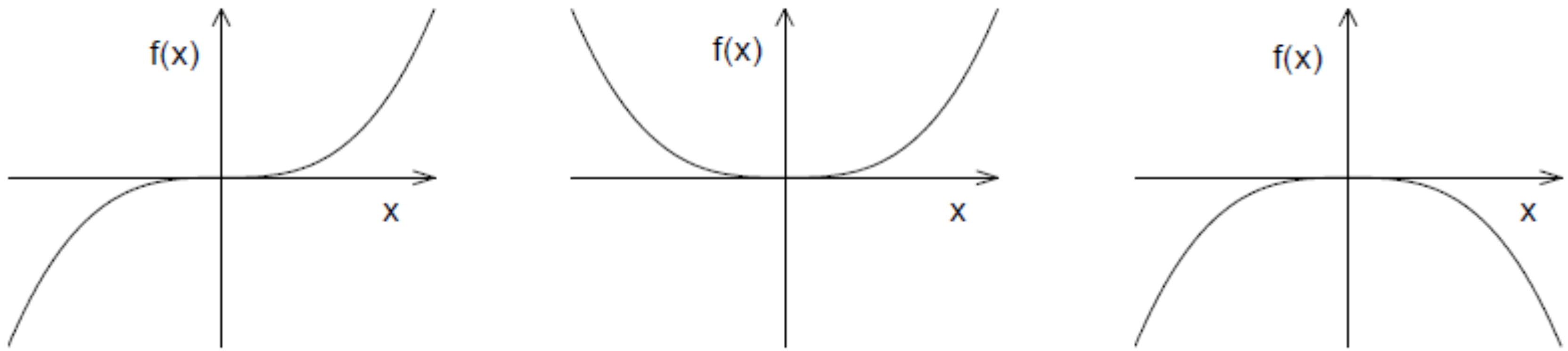
- unstable if it is not stable
- asymptotically stable if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

Unstable Scalar Systems

$$\dot{x} = f(x)$$

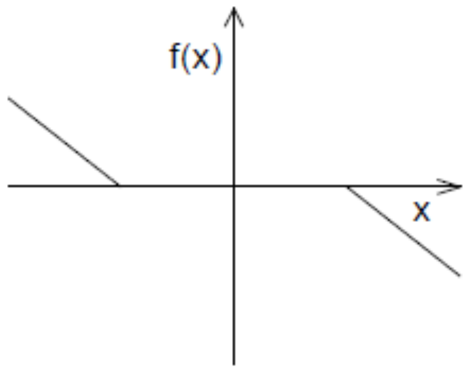
The $\varepsilon - \delta$ requirement for stability is violated if $xf(x) > 0$ on either side of the origin. Why?



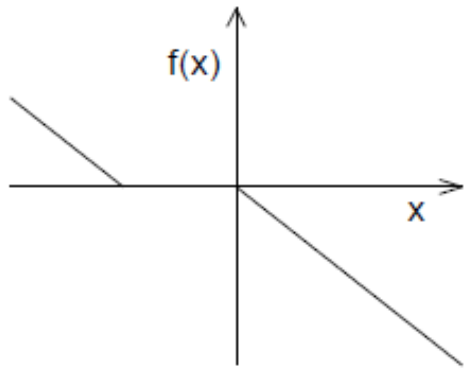
Unstable Unstable Unstable

Stable Scalar Systems

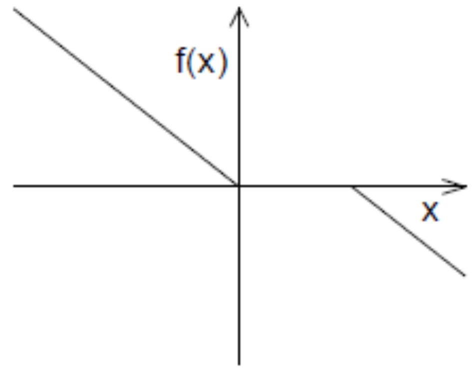
The origin is stable if and only if $xf(x) \leq 0$ in some neighborhood of the origin



Stable



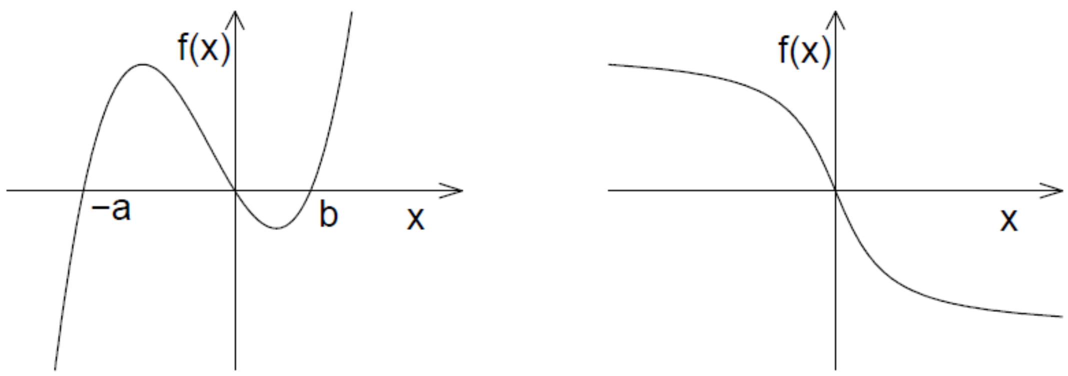
Stable



Stable

Asymptotically Stable Scalar Systems

The origin is asymptotically stable if and only if $xf(x) < 0$ in some neighborhood of the origin



(a) (b)

Asymptotically Stable Globally Asymptotically Stable

Region of attraction

$$\dot{x} = f(x), \qquad x(0) = x_0$$

Asymptotically stable if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

- The region of attraction (also called region of asymptotic stability, domain of attraction, or basin) is the set of all points x_0 in D such that the solution of is defined for all $t \geq 0$ and converges to the origin as $t \to \infty$.
- The origin is globally asymptotically stable if the region of attraction is the whole space R^n.

Linear Time-Invariant Systems

- The solution of $\dot{x} = Ax$ for a given initial state $x(0)$ is given by

$$x(t) = \exp(A\,t)x(0)$$

- The Jordan form of A is given by

$$P^{-1}AP = J = \text{block diag}[J_1, J_2, \ldots, J_r]$$

where J_i is a Jordan block associated with the eigenvalue λ_i of A.

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & \ldots & 0 \\ 0 & \lambda_i & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & \ldots & \ldots & \ldots & 0 & \lambda_i \end{bmatrix}$$

Linear Time-Invariant Systems

Therefore,

$$\exp(At) = P\exp(Jt)P^{-1} = \sum_{i=1}^{r}\sum_{k=1}^{m_\mathrm{i}} t^{k-1}\exp(\lambda_i t)R_{ik}$$

m_i is the order of the Jordan block J_i, r is the number of different eigenvalues, R_{ik} are constant matrices.

- $\mathrm{Re}[\lambda_i] < 0 \quad \forall i \quad \Leftrightarrow$ Asymptotically Stable
- $\mathrm{Re}[\lambda_i] > 0$ for some i $\Rightarrow$ Unstable
- $\mathrm{Re}[\lambda_i] \leq 0 \;\; \forall i$ & $m_i > 1$ for $\mathrm{Re}[\lambda_i] = 0 \;\Rightarrow$ Unstable
- $\mathrm{Re}[\lambda_i] \leq 0 \;\; \forall i$ & $m_i = 1$ for $\mathrm{Re}[\lambda_i] = 0 \;\Rightarrow$ Stable

Linear Time-Invariant Systems

Theorem

- The equilibrium point $x = 0$ of $\dot{x} = Ax$ is stable if and only if all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] \leq 0$ and for every eigenvalue with $\mathrm{Re}[\lambda_i] = 0$ and algebraic multiplicity $q_i \geq 2$, $\mathrm{rank}(A - \lambda_i I) = n - q_i$, where n is the dimension of x.
- The equilibrium point $x = 0$ is globally asymptotically stable if and only if all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] < 0$. When all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] < 0$, A is called a Hurwitz matrix.

Exponential Stability

Definition

- The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is exponentially stable if

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \qquad \forall \mathrm{t} \geq 0$$

  for some $k \geq 1, \lambda > 0$, for all $\|x(0)\| < c$.

- It is globally exponentially stable if the inequality is satisfied for any initial state $x(0)$.

- Exponential Stability ⇒ Asymptotic Stability
- For linear systems, asymptotic stability is equivalent to exponential stability

Example

$$\dot{x} = -x^3$$

The origin is asymptotically stable

$$x(t) = \frac{x(0)}{\sqrt{1 + 2tx^2(0)}}$$

It is NOT exponentially stable. Why? Because

$$|x(t)| \le k\mathrm{e}^{-\lambda t}|x(0)| \Rightarrow \frac{\mathrm{e}^{2\lambda t}}{1 + 2tx^2(0)} \le k^2$$

This is impossible because $\lim_{t\to\infty} \frac{\mathrm{e}^{2\lambda t}}{1+2tx^2(0)} = \infty$

Linearization of a Nonlinear System

- Consider the n-dimensional system

$$\dot{x} = f(x)$$

where f is a continuously differentiable function over D=$\{\|x\| < r\}$ for some $r = 0$, and $f(0) = 0$.

- Let $J(x)$ be the Jacobian matrix of $f(x)$

$$J(x) = \frac{\partial f}{\partial x}(x)$$

Additionally, define

$$h(\sigma) = f(\sigma x) \text{ for } 0 \leq \sigma \leq 1, \qquad h'(\sigma) = J(\sigma x)x$$

$$h(1) - h(0) = \int_0^1 h'(\sigma)\, \mathrm{d}\sigma, \qquad h(0) = f(0) = 0$$

$$f(x) = \int_0^1 J(\sigma x)\, \mathrm{d}\sigma\, x$$

Linearization

- Set $A = J(0)$ and add and subtract Ax, yielding

$$f(x) = [A + G(x)]x$$

where $G(x) = \int_0^1 [J(\sigma x) - J(0)]\, \mathrm{d}\sigma$, $G(x) \to 0$ as $x \to 0$

- This suggests that in a small neighborhood of the origin we can approximate the nonlinear system $\dot{x} = f(x)$ by its linearization about the origin $\dot{x} = Ax$.

Linearization

$$\dot{x} = f(x) \qquad \text{nonlinear system}$$

$$\dot{x} = Ax \qquad \text{linearized system}$$

Theorem

- The origin is exponentially stable if and only if $\mathrm{Re}[\lambda_i] < 0$ for all eigenvalues of A.
- The origin is unstable if $\mathrm{Re}[\lambda_i] > 0$ for some i.

- Linearization fails when $\mathrm{Re}[\lambda_i] \leq 0$ for all i, with $\mathrm{Re}[\lambda_i] = 0$ for some i.

Example

- Check the stability of the scalar system

$$\dot{x} = (x-1)^3$$

- Linearizing the system about the origin yields

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = 3(x-1)^2\Big|_{x=0} = 3$$

- The system is unstable.