Control Systems Stability and Robust Control

Lecture #3

Lyapunov Theorems for Stability

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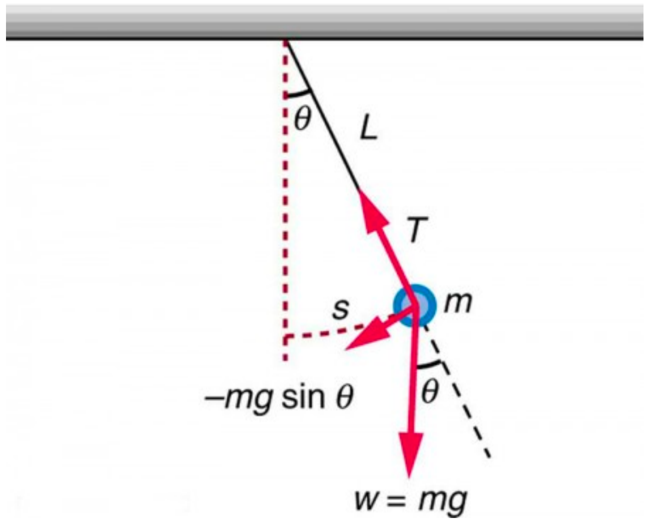
A motivating example

- Consider the pendulum equation

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a \sin x_1 - b \sin x_2$$



- It is difficult to find its all solutions. However, the conclusions about the stable equilibrium point of the pendulum can be reached by using energy concepts.

A motivating example

- Let us define the energy of the pendulum $E(x)$ as the sum of its potential and kinetic energies:

$$E(x) = \int_0^{x_1} a \sin y \, \mathrm{d}y + \frac{1}{2} x_2^2 = a(1 - \cos x_1) + \frac{1}{2} x_2^2$$

- $E(0) = 0$
- When friction is accounted for ($b > 0$), energy will dissipate during the motion of the system, that is, $\mathrm{d}E/\mathrm{d}t \leq 0$, showing that the trajectory tends to $x = 0$ as t tends to ∞. Therefore, the system is asymptotically stable.

Lyapunov's Method

$$\dot{x} = \mathrm{f}(x)$$

- Let $V\ (x)>0$ for all $x \neq 0$ be a continuously differentiable defined in a domain $\mathrm{D} \subset R^n$; $0 \in \mathrm{D}$.
- The derivative of V along the trajectories of is:

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x)$$

- We use this idea to determine the stability of the system, which is called Lyapunov's Method. The energy function is called Lyapunov function.

Lyapunov Theorem

- If there is $V(x)$ *such that*

$$V(0) = 0,$$
$$V(x) > 0, \forall x \in D \text{ with } x \neq 0,$$
$$\dot{V}(x) \leq 0, \forall x \in D$$

then the origin is a stable.

- Moreover, if

$$\dot{V}(x) < 0, \forall x \in D \text{ with } x \neq 0$$

then the origin is asymptotically stable.

- Furthermore, if

$$V(x) > 0, \forall x \neq 0,$$
$$\|x\| \to \infty \Rightarrow V(x) \to \infty, \qquad \text{(radically unbounded)}$$
$$\dot{V}(x) < 0, \forall x \neq 0$$

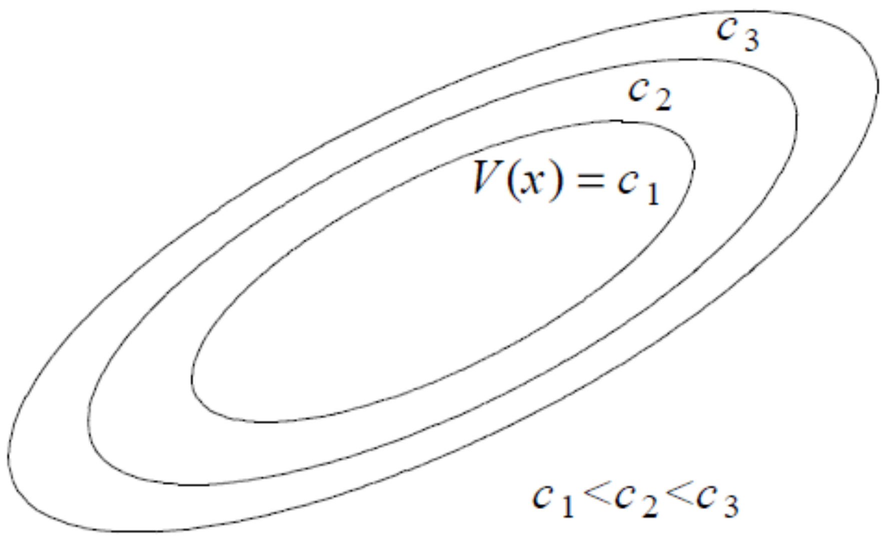
then the origin is globally asymptotically stable.

Terminology of Lyapunov Function

$V(0) = 0, V(x) \geq 0 \; for \; x \neq 0$	Positive semidefinite
$V(0) = 0, V(x) > 0 \; for \; x \neq 0$	Positive definite
$V(0) = 0, V(x) \leq 0 \; for \; x \neq 0$	Negative semidefinite
$V(0) = 0, V(x) < 0 \; for \; x \neq 0$	Negative definite

Level Surface and Level Set

- The set $\{x \mid V(x) = c\}$, for some $c > 0$, is called a Lyapunov surface or a level surface.
- The set $\{x \mid V(x) \leq c\}$, for some $c > 0$, is called a level set.

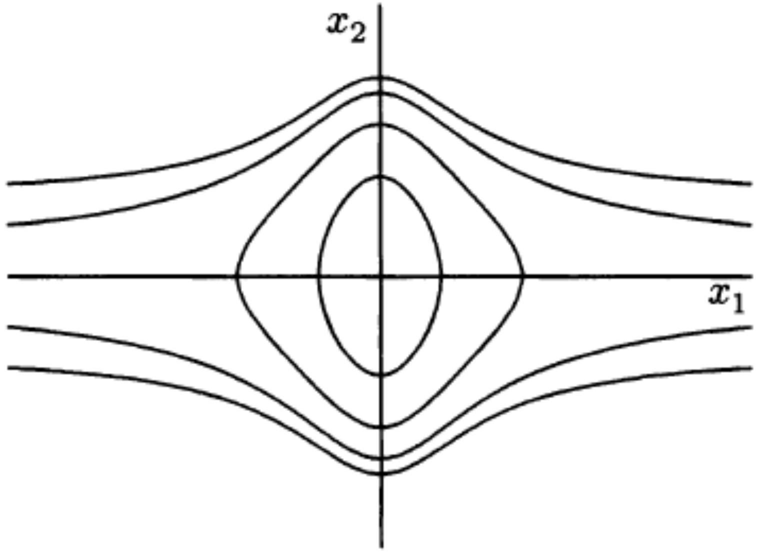


Example

- Consider the function:

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$$

where Ω_c=$\{x\epsilon R^n | V(x) \leq c\}$ and the right plot corresponds to the surfaces $\{x \in R^n | V(x) = c\}$.



- Whether $V(x)$ is radically unbounded?
- For small c, the surface $\{x \in R^n | V(x) = c\}$ is closed; hence, Ω_c is bounded since it is contained in a closed ball B_r for some $r > 0$.
- As c increases, a value is reached after which the surface $V(x) = c$ is open and Ω_c is unbounded.

Necessity of the existence of Lyapunov Functions

- Whether the conditions of Lyapunov's theorem is necessary?
- Failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that the equilibrium point is not stable or asymptotically stable. It only means that such stability property cannot be established by using this Lyapunov function candidate.

Example

- Recall the pendulum equation

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a \ \sin x_1 - b \sin x_2$$

- V(x) $= a(1 - \cos x_1) +$ (1/2)x_2^2 as a Lyapunov function candidate.

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2 \, \dot{x}_2 = \text{-}b \ x_2^2$$

- When b>0,the derivative $\dot{V}(x)$ is negative semidefinite. It is not negative definite because V(x) $=$ 0 for $x_2 =$ 0 irrespective of the value of x_1; that is, $\dot{V}(x) =$ 0 along the x_1-axis. Therefore, we can only conclude that the origin is stable. However, in fact when b>0,the origin is asymptotically stable.

Example

- Now we try

$$V(x) = \frac{1}{2}x^T P x + a(1 - cos x_1)$$
$$= \frac{1}{2}[x_1\ x_2]\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + a(1 - cos x_1)$$
$$p_{11} > 0, \qquad p_{11}p_{22} - p_{12}^2 > 0$$

- The derivative of the function is

$$\dot{V}(x) = (p_{11}x_1 + p_{12}x_2 + a sin x_1)x_2$$
$$+(p_{12}x_1 + p_{22}x_2)(-a\sin x_1 - bx_2)$$
$$= \mathrm{a}(1 - p_{22})x_2 \sin x_1 - ap_{12}x_1 sin x_1$$
$$+(p_{11} - p_{12}b)x_1x_2 + (p_{12} - p_{22}b)x_2^2$$

Example

- Take

$$p_{22} = 1, p_{11} = b, p_{12} = p_{21} = \frac{b}{2}$$

- We have

$$\dot{V}(x) = -\frac{1}{2}abx_1\sin x_1 - \frac{1}{2}bx_2^2$$

$$D = \{x \mid |x_1| < \pi\}$$

- When b>0, $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite over D. The origin is asymptotically stable.