Control Systems Stability and Robust Control

Lecture #4

Estimates of region of attraction

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Region of attraction

$$\dot{x} = f(x), \qquad x(0) = x_0$$

Asymptotically stable if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

- The region of attraction (also called region of asymptotic stability, domain of attraction, or basin) is the set of all points x_0 in D such that the solution of is defined for all $t \geq 0$ and converges to the origin as $t \to \infty$.
- The origin is globally asymptotically stable if the region of attraction is the whole space R^n.

How does a region of attraction look like?

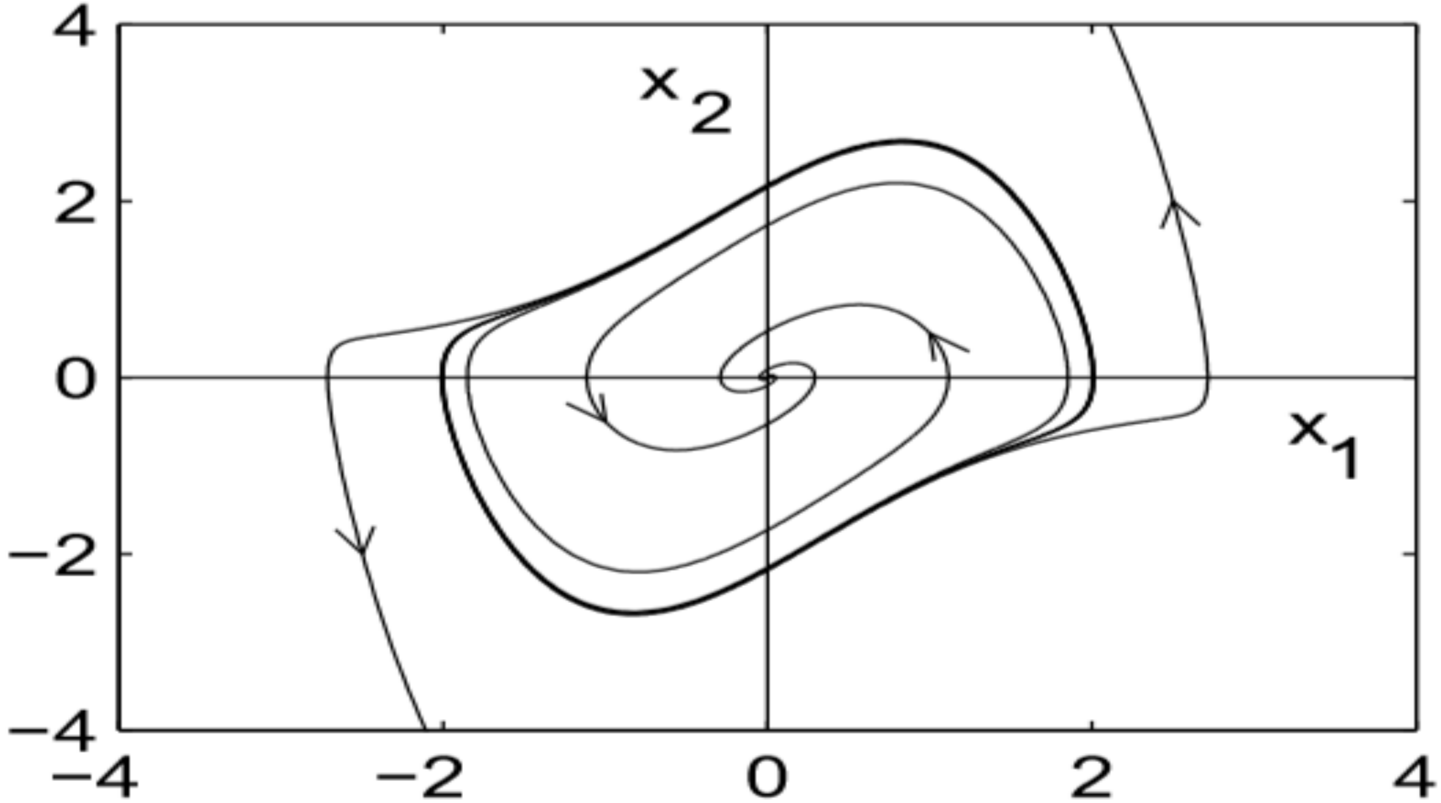
Lemma

The region of attraction of an asymptotically stable equilibrium point is an open, connected, invariant set, and its boundary is formed by trajectories.

An example

Consider the two-dimensional system

$$\dot{x}_1 = -x_2, \qquad \dot{x}_2 = x_1 + (x_1^2 - 1)x_2$$



Where is the region of attraction of the above system?

Estimates of the region of attraction

- Let D be a domain with $0 \in D$ such that for all $x \in D$, $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite. Is D a subset of the region of attraction?

No. Because D might not be positively invariant.

Example

- Consider the following system

$$\dot{x}_1 = -x_2, \qquad \dot{x}_2 = -x_1 + \frac{1}{3}x_1^3 - x_2$$

- Define the Lyapunov function

$$V(x) = \frac{1}{2}x^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + 2\int_0^{x_1} \left(y - \frac{1}{3}y^3\right) dy$$

$$= \frac{3}{2}x_1^2 - \frac{1}{6}x_1^4 + x_1 x_2 + x_2^2$$

$$\dot{V}(x) = -x_1^2\left(1 - \frac{1}{3}x_1^2\right) - x_2^2$$

$$D = \{-\sqrt{3} < x_1 < \sqrt{3}\}$$

Is D a subset of the region of attraction?

Level set as the estimate

- By LaSalle's Invariance Principle, if D is a domain that contains the origin such that $\dot{V}(x) \le 0$ in D, then the region of attraction can be estimated by a compact positively invariant set $\Gamma \in D$ if
  - $\dot{V}(x) < 0$ for all $x \in \Gamma, x \neq 0$, or
  - No solution can stay identically in $\{x \in D \mid \dot{V}(x) = 0\}$ other than the zero solution.
- The simplest such estimate is the set $\Omega_c = \{V(x) \le c\}$ when Ω_c is bounded and contained in D.

Level set of quadratic Lyapunov functions

- Quadratic Lyapunov function

$$V(x) = x^T P x, \qquad P = P^T, \qquad \Omega_c = \{V(x) \leq c\}$$

- Suppose that $D = \{||x|| < r\}$, find a value of c such that $\Omega_c \subseteq D$?
- If $D = \{||x|| < r\}$, then $\Omega_c \subseteq D$ if

$$c < \min_{||x||=r} x^T P x = \lambda_{\min}(P) r^2$$

Example

- Consider the system

$$\dot{x}_1 = -x_2, \qquad \dot{x}_2 = x_1 + (x_1^2 - 1)x_2$$

$$A = \frac{\partial f}{\partial x}\Big|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

- Because A is Hurwitz, so the system has an asymptotically stable origin.

- Select $P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$, with $\lambda_{min}(P) = 0.691$

Example- Cont'd

- Let

$$V(x) = x^T P x = 1.5x_1^2 - x_1 x_2 + x_2^2$$

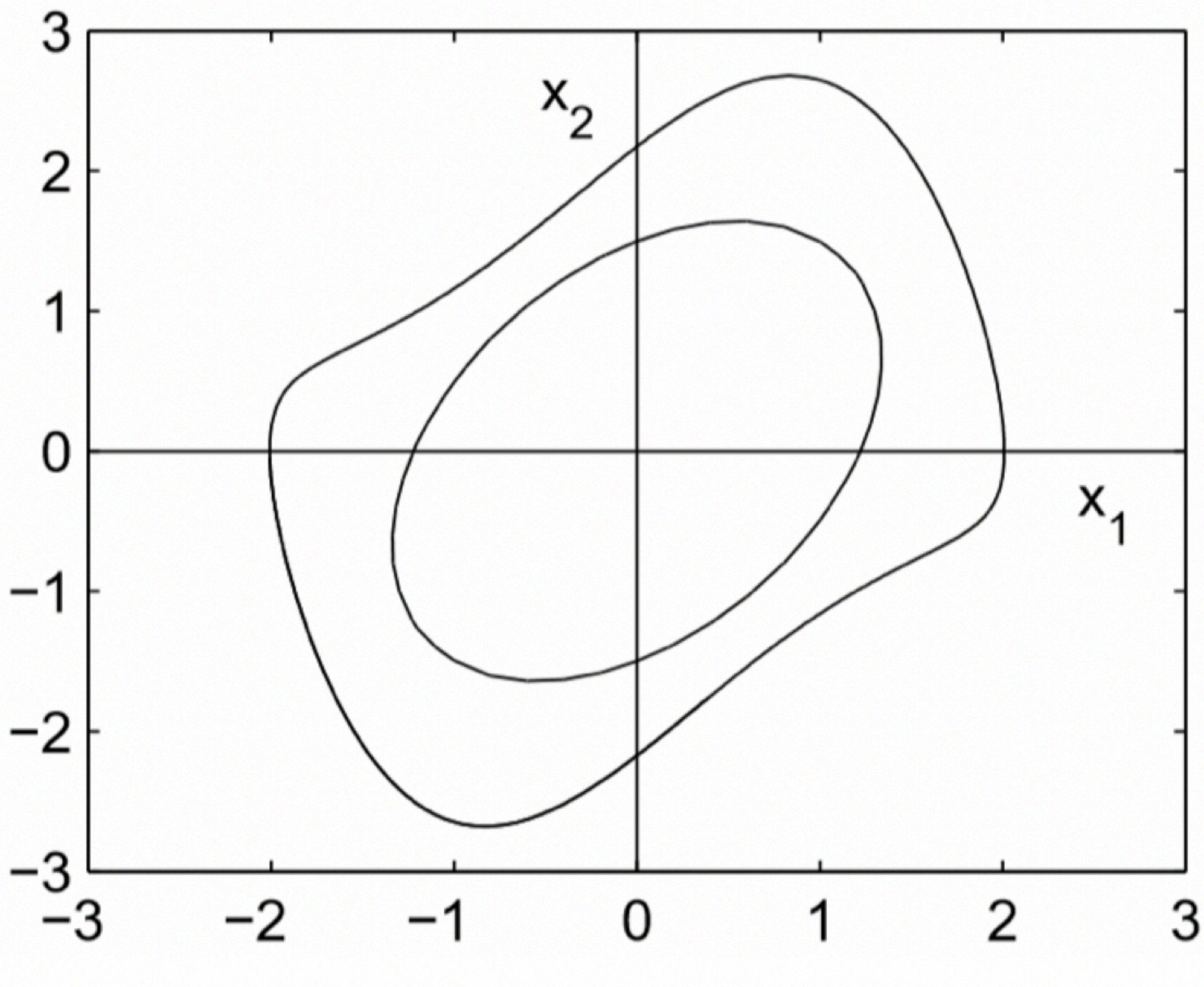
$$\dot{V}(x) = -(x_1^2 + x_2^2) - x_1^2 x_2 (x_1 - x_2)$$

- The derivative of $V(x)$ is given by

$$\dot{V}(x) \le -\|x\|^2 + \frac{\sqrt{5}}{2}\|x\|^4 < 0 \text{ for } 0 < \|x\|^2 < \frac{2}{\sqrt{5}} = r^2$$

- Take $c < \lambda_{min}(P) r^2 = 0.691 \times \frac{2}{\sqrt{5}} = 0.618$
- $\{x \mid V(x) \le c\}$ is an estimate of the region of attraction.

Example- Cont'd



Multiple estimates

- If $\Omega_1, \Omega_2, \dots, \Omega_m$ are positively invariant subsets of the region of attraction, then their union $\cup_{i=1}^{n} \Omega_i$ is also a positively invariant subset of the region of attraction.
- Therefore, we have multiple Lyapunov functions for the same system and each function is used to estimate the region of attraction, we can enlarge the estimate by taking the union of all the estimates.

Converse Lyapunov theorem for asymptotic stability

Theorem

Let $x = 0$ be an asymptotically stable equilibrium point for $\dot{x} = f(x)$, where f is locally Lipschitz on a domain $D \subset R^n$ that contains the origin. Let $R_A \subset D$ be the region of attraction of $x = 0$. Then, there is a smooth, positive definite function $V(x)$ and a continuous, positive definite function $W(x)$, both defined for all $x \in R_A$, such that

$$V(x) \to \infty \text{ as } x \to \partial R_A$$

$$\frac{\partial V}{\partial x} f(x) \leq -W(x), \qquad \forall x \in R_A$$

and for any $c > 0$, $\{V(x) \leq c\}$ is a compact subset of R_A. When $R_A = R^n$, $V(x)$ is radially unbounded.

Converse Lyapunov theorem for exponential stability

Let $x = 0$ be an exponential stable equilibrium point for the system $\dot{x} = f(x)$, where f is continuously differentiable on $D = \{||x|| < r\}$. Let $k, \lambda,$ and r_0 be positive constants with $r_0 < \frac{r}{k}$ such that

$$||x(t)|| \leq k||x(0)||e^{-\lambda t}, \ \forall x(0) \in D_0, \ \forall t \geq 0$$

where $D_0 = \{||x|| < r_0\}$. Then, there is continuously differentiable function $V(x)$ that satisfies the inequalities

$$c_1||x||^2 \leq V(x) \leq c_2||x||^2, \qquad \frac{\partial V}{\partial x} f(x) \leq -c_3||x||^2$$

$$\left\|\frac{\partial V}{\partial x}\right\| \leq c_4||x||$$

for all $x \in D_0$, with positive constants $c_1, c_2, c_3,$ and c_4. Moreover, if f is continuously differentiable for all x, globally Lipschitz, and the origin is globally exponentially stable, then $V(x)$ is defined and satisfies the aforementioned inequalities for all $x \in R^n$.