Control Systems Stability and Robust Control

Lecture #5

Time-Varying Systems

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Time-varying Systems

- Consider the n-dimensional time-varying system

$$\dot{x} = f(t, x)$$

$f(t, x)$ is piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and all $x \in D$, $(0 \in D)$.

- The origin is an equilibrium point if

$$f(t, 0) = 0, \qquad \forall t \geq 0$$

- The solution of the time-varying system depends on t_0.

Class K functions

- A scalar continuous function $\alpha(r)$, defined for $r \epsilon [0, a)$, belongs to class K if it is strictly increasing and $\alpha(0) = 0$.
- It belongs to class K_∞ if it is defined for all $r \geq 0$ and $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$.

Examples of class K functions

- $\alpha(r) = \tan^{-1}(r)$

  $\alpha'(r) = \frac{1}{(1+r^2)} > 0 \rightarrow$ strictly increasing $\rightarrow$ class K.

  $\lim_{r\to\infty} \alpha(r) = \frac{\pi}{2} < \infty$ $\rightarrow$ it does not belong to class K_∞

- $\alpha(r) = r^c, c > 0$

  $\alpha'(r) = cr^{c-1}(x) > 0 \rightarrow$ strictly increasing $\rightarrow$ class K.

  $\lim_{r\to\infty} \alpha(r) = \infty \rightarrow$ class K_∞.

- $\alpha(r) = \min\{r, r^2\}$ is continuous, strictly increasing, $\lim_{r\to\infty} \alpha(r) = \infty$; it belongs to class K_∞. It is not continuously differentiable at $r = 1$. Continuous differentiability is not required for a class K function.

Class KL functions

A scalar continuous function $\beta(r, s)$, defined for $r \in [0, a)$ and $s \in [0, \infty)$, belongs to class KL if,

- for each fixed s, the mapping $\beta(r, s)$ belongs to class K with respect to r ,
- for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s, and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$.

Examples of class KL functions

- $\beta(r,s)=\frac{r}{ksr+1}, \forall k>0$,

$$\frac{\partial\beta}{\partial r}=\frac{1}{(ksr+1)^2}>0 \text{ strictly increasing in } r$$

$$\frac{\partial\beta}{\partial s}=\frac{-kr^2}{(ksr+1)^2}<0 \text{ strictly decreasing in } s.$$

$\beta(r,s)\to 0$ as $s\to\infty$. It belongs to class $K\mathcal{L}$.

- $\beta(r,s)=r^c e^{-as}, \forall a>0, c>0$ whether $\beta(r,s)$ belongs to class $K\mathcal{L}$?

$$\frac{\partial\beta}{\partial r}=cr^{c-1}e^{-as}>0 \text{ strictly increasing in } r$$

$$\frac{\partial\beta}{\partial s}=-ar^c e^{-as}<0 \text{ strictly decreasing in } s.$$

$\beta(r,s)\to 0$ as $s\to\infty$. It belongs to class $K\mathcal{L}$.

Some results on class K and KL functions

Let α_1 and α_2 be class K function on $[0, a_1)$ and $[0, a_2)$, respectively, with $a_1 \geq \lim_{r\to a_2} \alpha_2(r)$,and β be a class $K\mathcal{L}$ function defined on $[0, \lim_{r\to a_2} \alpha_2(r))\times[0,\infty)$ with $a_1 \geq \lim_{r\to a_2} \beta(\alpha_2(r), 0)$. Let α_3 and α_4 be class K_∞ functions. Denote the inverse of α_i by α_i^{-1}. Then,

- α_1^{-1} is defined on $[0, \lim_{r\to a_1} \alpha_1(r))$, and belongs to class K.
- α_3^{-1} is defined on $[0,\infty)$ and belongs to class K_∞.
- $\alpha_1 \circ \alpha_2$ is defined on $[0, \alpha_2)$ and belongs to class K
- $\alpha_3 \circ \alpha_4$ is defined on $[0,\infty)$ and belongs to class K_∞
- $\sigma(r,s) = \alpha_1\big(\beta(\alpha_2(r), s)\big)$ is defined on $[0, a_2)\times [0,\infty)$ and belongs to class $K\mathcal{L}$

Class K functions as bounds on positive definite functions

- Let $V: D \rightarrow R$ be a continuous positive definite function defined on a doma in $D \subset R^n$ that contains the origin. Let $B_r \subset D$ for some $r > 0$.Then, Th ere exist class K functions $\alpha_1$and α_2 , defined on $[0, r]$, such that

$$\alpha_1(||x||) \leq V(x) \leq \alpha_2(||x||)$$

- For all $x \in B_r$. If $D = R^n$ and $V(x)$ is radially unbounded, then there exi st class K_∞ functions α_1 and α_2 such that the foregoing inequality holds f or all $x \in R^n$

Stability notions for time-varying systems

The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is

- uniformly stable if there exist a class K function α and a positive constant c, independent of t_0, such that

$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \qquad \forall t \geq t_0 \geq 0, \forall \|x(t_0)\| < c$$

- uniformly asymptotically stable if there exist a class $K\mathcal{L}$ function β and a positive constant c, independent of t_0, such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \qquad \forall t \geq t_0 \geq 0, \forall \|x(t_0)\| < c$$

- globally uniformly asymptotically stable if the foregoing inequality holds $\forall x(t_0)$

Stability notions for time-varying systems- Cont'd

- exponentially stable if there exist positive constants c, k and λ such that

$$||x(t)|| \leq k||x(t_0)||e^{-\lambda(t-t_0)}, \forall \ ||x(t_0)|| < c$$

- globally exponentially stable if the foregoing inequalit $\forall x(t_0)$

Lyapunov theorem for uniform stability

Theorem

Let the origin $x = 0$ be an equilibrium point of $\dot{x} = f(t,x)$ and $D \subset R^n$ be a domain containing $x = 0$. Suppose $f(t,x)$ is piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and $x \in D$.Let $V(t,x)$ be a continuously differentiable function such that

$$W_1(x) \leq V(t,x) \leq W_2(x)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0$$

for all $t \geq 0$ and $x \in D$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on D. Then, the origin is uniformly stable.

Lyapunov theorem for uniform asymptotic stability

Theorem

Suppose the assumptions of the previous theorem are satisfied with

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -W_3(x)$$

for all $t \ge 0$ and $x \in D$, where $W_3(x)$ is a continuous positive definite functi on on D. Then, the origin is uniformly asymptotically stable. Moreover, if r and c are chosen such that $B_r = \{\|x\| \le r\} \subset D$ and $c < \min_{\|x\|=r} W_1(x)$, the n every trajectory starting in $\{W_2(x) \le c\}$ satisfies

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \qquad \forall t \ge t_0 \ge 0$$

Lyapunov theorem for globally uniformly asymptotic stability

Theorem

Finally, if $D = R^n$ and $W_1(x)$ is radially unbounded, then the origin is globally uniformly asymptotically stable.

Lyapunov theorem for exponential stability

Theorem

Suppose the assumptions of the previous theorem are satisfied with

$$k_1\|x\|^a \leq V(t,x) \leq k_2\|x\|^a$$

$$\dot{\boldsymbol{V}}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -k_3\|x\|^a$$

for all $t \geq 0$ and $x \in D$, where k_1, k_2, k_3, and a are positive constants. Then, the origin is exponentially stable. If the assumptions hold globally, the origin will be globally exponentially stable.

Example

Consider the scalar system

$$\dot{x} = -[1 + g(t)]x^3, g(t) \geq 0, \forall t \geq 0$$

$$V(x) = \frac{1}{2}x^2$$

where $\mathrm{g}(t)$ is a continuous. We obtain

$$\dot{V}(t, x) = -[1 + g(t)]x^4 \leq -x^4$$

$$W_3(x) = x^4, W_1(x) = W_2(x) = V(x)$$

The origin is globally uniformly asymptotically stable