# Control Systems Stability and Robust Control

## Lecture #7

## Boundedness and Ultimate Boundedness

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# Boundedness and Ultimate Boundedness

The solution of $\dot{x} = f(t, x)$ is

- uniformly bounded if there exists $c > 0$, independent of $t_0$, and for every $a \in (0, c)$, there is $\beta > 0$, dependent on $a$ but independent of $t_0$, such that

$$\|x(t_0)\| \le a \Rightarrow \|x(t)\| \le \beta, \forall\, t \ge t_0$$

- globally uniformly bounded if the upper formula holds for arbitrarily large $a$.

# Boundedness and Ultimate Boundedness

- uniformly ultimately bounded with ultimate bound $b$ if there exists a positive constant $c$, independent of $t_0$, and for every $a \in (0, c)$, there is $T \geq 0$, dependent on $a$ and $b$ but independent of $t_0$, such that

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b, \forall t \geq t_0 + T$$

- globally uniformly ultimately bounded if the upper formula holds for arbitrarily large a.

# Lyapunov Theorem for Ultimate Boundedness

- Suppose $B_\mu \subset D \subset R^n$ and

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial x} f(t, x) \leq -W_3(x), \forall x \in D \text{ with } \|x\| \geq \mu, \forall t \geq 0$$

where $\alpha_1$ and $\alpha_2$ are class $\mathcal{K}$ functions and $W_3(x)$ is positive definite.

- Choose $c > 0$ such that $\Omega_c = \{V(x) \leq c\}$ is compact and contained in $D$ and $\mu < \alpha_2{}^{-1}(c)$.
- $\Omega_c$ is positively invariant and there exists a class $\mathcal{KL}$ function $\beta$ such that for every $x(t_0) \in \Omega_c$,

$$\|x(t)\| \leq max\{\beta(\|x(t_0)\|, t - t_0), \alpha_1{}^{-1}(\alpha_2(\mu))\}, \forall t \geq t_0$$

- If $D = R^n$ and $\alpha_1 \in \mathcal{K}_\infty$, the inequality holds $\forall x(t_0), \forall \mu$.

# Some remarks

- The ultimate bound is independent of the initial state.
- The ultimate bound is a class $\mathcal{K}$ function of $\mu$; hence, the smaller the value of $\mu$, the smaller the ultimate bound. As $\mu \longrightarrow 0$, the ultimate bound approaches zero.

# Example

- Consider the following system:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -(1 + {x_1}^2)x_1 - x_2 + M\cos\omega t\,, M \geq 0$$

- Define the following Lyapunov function

$$V(x) = x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + \frac{1}{2}{x_1}^4 \stackrel{\text{def}}{=} x^T P x + \frac{1}{2}{x_1}^4$$

- Can you use the Lyapunov theorem to show the ultimate bound of the system?

# Example- Cont'd

$$\lambda_{min}(P)\|x\|^2 \le V(x) \le \lambda_{max}(P)\|x\|^2 + \frac{1}{2}\|x\|^4$$

$$\alpha_1(r) = \lambda_{min}(P)r^2, \qquad \alpha_2(r) = \lambda_{max}(P)r^2 + \frac{1}{2}r^4$$

$$\dot{V} = -x_1{}^2 - x_1{}^4 - x_2{}^2 + (x_1 + 2x_2)M\cos\omega t$$

$$\le -\|x\|^2 - x_1{}^4 + M\sqrt{5}\|x\| \qquad (0 < \theta < 1)$$

$$= -(1-\theta)\|x\|^2 - x_1{}^4 - \theta\|x\|^2 + M\sqrt{5}\|x\|$$

$$\le -(1-\theta)\|x\|^2 - x_1{}^4, \qquad \forall \|x\| \ge M\sqrt{5}/\theta \stackrel{\text{def}}{=} \mu$$

The solutions are GUUB by

$$\alpha_1{}^{-1}(\alpha_2(\mu)) = \sqrt{\frac{\lambda_{max}(P)\mu^2 + \mu^4/2}{\lambda_{min}(P)}}$$

# Ultimate Boundedness via Quadratic Lyapunov Functions

- Suppose

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(t,x) \leq -c_3\|x\|^2, \forall\, x \in D \text{ with} \|x\| \geq \mu, \forall\, t \geq 0$$

for some positive constants $c_1$ to $c_3$, and $\mu < \sqrt{c/c_2}$.

- Then, $\Omega_c = \{V(x) \leq c\}$ is positively invariant and $\forall\, x(t_0) \in \Omega_c$

$$\|x(t)\| \leq \sqrt{\frac{c_2}{c_1}} \max\{\|x(t_0)\| e^{-\frac{\left(\frac{c_3}{c_2}\right)(t-t_0)}{2}}, \mu\}, \quad \forall\, t \geq t_0$$

- If $D = R^n$, the inequalities hold $\forall x(t_0), \forall \mu$.

# Perturbed Systems: Nonvanishing Perturbation

- Nominal System

$$\dot{x} = f(x), \qquad f(0) = 0$$

- Perturbed System

$$\dot{x} = f(x) + g(t, x), \qquad g(t, 0) \neq 0$$

- Case 1: The origin of $\dot{x} = f(x)$ is exponentially stable.
- Case 2: The origin of $\dot{x} = f(x)$ is asymptotically stable.

# Perturbed Systems

- Suppose that $\forall\, x \in B_r, \forall\, t \geq 0$

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x) \leq -c_3\|x\|^2, \qquad \left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\|$$

$$\|g(t,x)\| \leq \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\,\theta r, \qquad 0 < \theta < 1$$

- Then, for all $x(t_0) \in \{V(x) \leq c_1 r^2\}$

$$\|x(t)\| \leq \max\{k \exp[-\gamma(t - t_0)]\|x(t_0)\|, b\}, \qquad \forall t \geq t_0$$

$$k = \sqrt{\frac{c_2}{c_1}}, \qquad \gamma = \frac{(1-\theta)c_3}{2c_2}, \qquad b = \frac{\delta c_4}{\theta c_3}\sqrt{\frac{c_2}{c_1}}$$

# Perturbed Systems

Suppose that $\forall\, x \in B_r, \forall\, t \geq 0$

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \qquad \frac{\partial V}{\partial x} f(x) \leq -\alpha_3(\|x\|)$$

$$\left\|\frac{\partial V}{\partial x}(x)\right\| \leq k, \qquad \|g(t,x)\| \leq \delta \leq \frac{\theta \alpha_3(\alpha_2{}^{-1}(\alpha_1(r)))}{k}$$

$\alpha_i \in K,\ 0 < \theta < 1$. Then, $\forall\, x(t_0) \in \{V(x) \leq \alpha_1(r)\}$

$$\|x(t)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \rho(\delta)\}, \forall\, t \geq t_0, \beta \in K£$$

$$\rho(\delta) = \alpha_1{}^{-1}\left(\alpha_2\left(\alpha_3{}^{-1}\left(\frac{\delta k}{\theta}\right)\right)\right)$$