Control Systems Stability and Robust Control

Lecture #8

Input-to-State Stability

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Input-to-State Stability (ISS)

Definition

The system $\dot{x} = f(x, u)$ is input-to-state stable if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial state $x(t_0)$ and any bounded input $u(t)$

$$\|x(t)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma(sup_{t_0 \leq \tau \leq t} \|u(\tau)\|)\}$$

for $\forall\, t \geq t_0$.

ISS of $\dot{x} = f(x, u)$ implies that

- BIBS stability
- $x(t)$ is ultimately bounded by $\gamma(sup_{t_0 \leq \tau \leq t} \|u(\tau)\|)$
- $lim_{t \to \infty} u(t) = 0 \Rightarrow lim_{t \to \infty} x(t) = 0$
- The origin of $\dot{x} = f(x, 0)$ is GAS

ISS Lyapunov Theorem

Theorem

Let $V\ (x)$ be a continuously differentiable function

$$\alpha_1(\|x\|) \leq\ V\ (x)\ \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial x}\, f(x,u) \leq -W_3(x), \qquad \forall\ \|x\|\ \geq\ \rho(\|u\|)\ >\ 0$$

$\forall\ x\ \in R^n,\ u\ \in R^m,$ where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty,\ \rho\ \in \mathcal{K},$ and $W_3(x)$ is a continuous positive definite function. Then, the system $\dot{x} =\ f(x,u)$ is ISS with $\gamma\ =\ {\alpha_1}^{-1} \circ \alpha_2\ \circ\ \rho.$

Exponentially stability -> ISS

Lemma

Suppose $f(x, u)$ is continuously differentiable and globally Lipschitz in (x, u). If $\dot{x} = f(x, 0)$ has a globally exponentially stable equilibrium point at the origin, then the system $\dot{x} = f(x, u)$ is input-to-state stable.

Example

Consider the system

$$\dot{x} = -x^3 + u$$

The origin of $\dot{x} = -x^3$ is globally asymptotically stable. Define the Lyapunov function

$$V = \frac{1}{2}x^2$$

The derivative is given by

$$\begin{aligned}\dot{V} &= -x^4 + xu \\ &= -(1-\theta)x^4 - \theta x^4 + xu \\ &\leq -(1-\theta)x^4, \quad \forall\, |x| \geq \left(\frac{|u|}{\theta}\right)^{1/3}, 0 < \theta < 1\end{aligned}$$

The system is ISS with $\gamma(r) = \left(\frac{r}{\theta}\right)^{1/3}$

Example

Show that the following system is ISS.

$$\dot{x} = -x - 2x^3 + (1 + x^2)\, u^2$$

Define the Lyapunov function

$$V = \frac{1}{2} x^2$$

The derivative is given by

$$\begin{aligned} \dot{V} &= -x^2 - 2x^4 + x(1 + x^2)u^2 \\ &= -x^4 - x^2(1 + x^2) + x(1 + x^2)u^2 \\ &\leq -x^4, \quad \forall\, |x| \geq u^2 \end{aligned} \qquad 0 < \theta < 1$$

The system is ISS with $\gamma(r) = r^2$

ISS for Cascade Systems

- If the systems $\dot{\eta} = f_1(\eta, \xi)$ and $\dot{\xi} = f_2(\xi, u)$ are input-to-state stable, then the cascade connection

$$\dot{\eta} = f_1(\eta, \xi), \qquad \dot{\xi} = f_2(\xi, u)$$

is input-to-state stable.

- Consequently, If $\dot{\eta} = f_1(\eta, \xi)$ is input-to-state stable and the origin of $\dot{\xi} = f_2(\xi)$ is globally asymptotically stable, then the origin of the cascade connection

$$\dot{\eta} = f_1(\eta, \xi), \qquad \dot{\xi} = f_2(\xi)$$

is globally asymptotically stable.

Example

- Consider the system:

$$\dot{x}_1 = -x_1 + {x_2}^2, \qquad \dot{x}_2 = -x_2 + u$$

- The system $\dot{x}_1 = -x_1 + {x_2}^2$ is input-to-state stable, as seen from Theorem 4.6 with

$$V\,(x_1) = \frac{1}{2}{x_1}^2.$$

$$\dot{V} = -{x_1}^2 - x_1{x_2}^2 \leq -(1-\theta){x_1}^2, \text{for} |x_1| \geq \frac{{x_2}^2}{\theta}, 0 < \theta < 1$$

- The linear system $\dot{x}_2 = -x_2 + u$ is input-to-state stable by Lemma 4.5.
- The cascade connection is input-to-state stable