Nonlinear Control
Lecture # 9
Positive Realness, Passivity, Stability

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Positive Real Transfer Functions

Definition

An $m \times m$ proper rational transfer function matrix G(s) is positive real if

1. poles of all elements of G(s) are in Re[s] ≤ 0.
2. for all real ω for which $j\omega$ is not a pole of any element of G(s), the matrix $G(j\omega) + G^T(-j\omega)$ is positive semidefinite.
3. any pure imaginary pole $j\omega$ of any element of G(s) is a simple pole and the residue matrix $\lim_{s \to j\omega} (s - j\omega) G(s)$ is positive semidefinite Hermitian.

- G(s) is strictly positive real if G(s − ε) is positive real for some ε > 0.

Scalar Case $m = 1$

- When $m = 1$, the following holds

$$\mathrm{G}(j\omega) + \mathrm{G}^T(-j\omega) = 2\mathrm{Re}[\mathrm{G}(j\omega)]$$

- The second condition of the definition reduces to

$$\mathrm{Re}[\mathrm{G}(j\omega)] \geq 0, \forall\ \omega \in [0, \infty)$$

which holds when the Nyquist plot of of $\mathrm{G}(j\omega)$ lies in the closed right-half complex plane.

Example for Positive Realness

Consider the transfer function

$$G(s) = \frac{1}{s}$$

has a simple pole at $s = 0$ whose residue is 1

$$\text{Re}[G(j\omega)] = \text{Re}[\frac{1}{jw}] = 0, \forall\ \omega \neq 0$$

Hence, G is positive real. It is not strictly positive real since

$$\frac{1}{(s - \varepsilon)}$$

has a pole in $\text{Re}[s] > 0$ for any $\varepsilon > 0$

A Lemma for Strictly Positive Realness

Lemma

An $m \times m$ proper rational transfer function matrix G(s) is strictly positive real if and only if

1. G(s) is Hurwitz
2. $G(j\omega) + G^T(-j\omega) > 0, \forall\ \omega \in R$
3. $G(\infty) + G^T(\infty) > 0$ or

$$\lim_{w \to \infty} \omega^{2(m-q)} \det[G(j\omega) + G^T(-j\omega)] > 0$$

where $q = \mathrm{rank}[G(\infty) + G^T(\infty)]$.

Scalar case $m = 1$

G(s) is strictly positive real if and only if

1. The transfer function $G(s)$ is Hurwitz.
2. $\mathrm{Re}[G(j\omega)] > 0, \forall\, \omega \in [0, \infty)$.
3. $G(\infty) > 0$ or $\lim_{w \to \infty} w^2 \mathrm{Re}[G(j\omega)] > 0$.

Example

- Show the system $G(s) = \frac{1}{s+a}$ $, a > 0$ is strictly positive real.

- Consider the system

$$G(s) = \frac{1}{s+a} \ , a > 0, \quad \text{is Hurwitz}$$

$$\mathrm{Re}[G(j\omega)] = \frac{a}{\omega^2 + a^2} > 0, \forall w \in [0, \infty)$$

$$\lim_{w \to \infty} w^2 \mathrm{Re}[G(j\omega)] = \lim_{w \to \infty} \frac{w^2 a}{\omega^2 + a^2} = a > 0 \Rightarrow \text{G is SPR}$$

Positive Realness of Linear Systems

Lemma

Let

$$G(s) = C(sI - A)^{-1} B + D$$

where (A,B) is controllable and (A,C) is observable. G(s) is positive real if and only if there exist matrices $P = P^T > 0$, L, and W such that

$$PA + A^T P = -L^T L$$

$$PB = C^T - L^T W$$

$$W^T W = D + D^T$$

Strictly Positive Realness of Linear Systems

Kalman–Yakubovich–Popov Lemma

Let

$$G(s) = C(sI - A)^{-1} B + D$$

where (A,B) is controllable and (A,C) is observable. G(s) is strictly positive real if and only if there exist matrices $P = P^T > 0$, L, and W, and a positive constant ε such that

$$PA + A^T P = -L^T L - \varepsilon P$$

$$PB = C^T - L^T W$$

$$W^T W = D + D^T$$

Positive Realness -> Passivity

Lemma

The linear time-invariant minimal realization

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

with

$$G(s) = C(sI - A)^{-1}B + D$$

is

(1) passive if G(s) is positive real.

(2) strictly passive if G(s) is strictly positive real.

Passivity -> Stability

Lemma

If the system

$$\dot{x} = f(x,u), y = h(x,u)$$

is passive with a positive definite storage function $V(x)$, then the origin of $\dot{x}$ $= f(x,0)$ is stable.

Proof

$$u^T y \geq \frac{\partial V}{\partial x} f(x,u) \Rightarrow \frac{\partial V}{\partial x} f(x,0) \leq 0$$

Example

Consider the system:

$$\dot{x} = f(x) + G(x)u, \qquad y = h(x), \qquad \dim(u) = \dim(y)$$

Suppose there is a positive definite V (x) such that

$$\frac{\partial V}{\partial x} f(x) \leq 0, \qquad \frac{\partial V}{\partial x} G(x) = h^T(x)$$

Since the system is passive

$$u^T y - \dot{V} = u^T h(x) - \frac{\partial V}{\partial x} f(x) - h^T(x) u = -\frac{\partial V}{\partial x} f(x) \geq 0$$

The origin of $\dot{x} = f(x)$ is stable.

Strict Passivity VS. Asymptotic Stability

Lemma

If the system

$$\dot{x} = f(x,u), \qquad y = h(x,u)$$

is strictly passive, then the origin of $\dot{x} = f(x,u)$ is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable.

Zero-State Observability

Definition

The system

$$\dot{x} = f(x,u), \qquad y = h(x,u)$$

is zero-state observable if no solution of $\dot{x} = f(x,0)$ can stay identically in $S = \{h(x,0) = 0\}$, other than the zero solution $x(t) \equiv 0$.

- Linear Systems

$$\dot{x} = Ax, \qquad y = Cx$$

- Observability of (A,C) is equivalent to

$$y(t) = Ce^{At}x(0) \equiv 0 \Leftrightarrow x(t) \equiv 0$$

Output Strict Passivity and Zero-State Observability

Lemma

If the system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is output strictly passive and zero-state observable, then the origin of $\dot{x} = f(x, 0)$ is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable.

Example

Consider the system:

$$\dot{x} = f(x) + G(x)u, \qquad y = h(x), \qquad \text{dim(u)} = \text{dim(y)}$$

Suppose there is a positive definite V (x) such that

$$\frac{\partial V}{\partial x} f(x) \leq -k h^T(x) h(x), \qquad \frac{\partial V}{\partial x} G(x) = h^T(x), k > 0$$

$$u^T y - \dot{V} \geq k y^T y$$

The system is output strictly passive. If, in addition, it is zero-state observable, then the origin of $\dot{x} = f(x)$ is asymptotically stable.

Example

Show the following system is globally asymptotically stable by output strict passivity.

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -ax_1^3 - kx_2 + u, \qquad y = x_2 \qquad a, k > 0$$

$$V(x) = \frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2$$

$$\dot{V}(x) = ax_1^3x_2 + x_2(-ax_1^3 - kx_2 + u) = -ky^2 + yu$$

The system is output strictly passive

$$y(t) \equiv 0 \Leftrightarrow x_2(t) \equiv 0 \Rightarrow ax_1^3(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

The system is zero-state observable. V is radially unbounded. Hence, the origin of the unforced system is globally asymptotically stable.